The XXZ model in the attractive regime with boundaries

Chaiho's home assignment

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- The inhomogenous XXZ model with boundaries
  $\rightarrow$ sG model with boundary interactions

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- Finite size properties of the lattice model

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    both repulsive and attractive regime through the NLIE approach

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  - Ahn- Nepomichie '03 , Ahn-Bellacosa-Ravanini '04 both repulsive and attractive regime through the NLIE approach
  - Fusion-TBA approach (Klümper-Pearce, Kuniba-Sakai-JS)
    well studied for the replusive regime
- Try the fusion approach in the attractive regime and see the consistency of the Ahn-Nepomichie result.

The outline of the talk

- The model and the scaling limit
- The fusion-TBA approach in the repulsive regime
- The fusion-TBA approach in the attractive regime
  - the origin of the difficulty
  - crucial property: Factorization by Q functions

The XXZ model with boundaries

- The Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\Big(\sum_{j=1}^{N-1}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z) + p\sigma_1^z + p'\sigma_N^z\Big)$$

- Parameterization

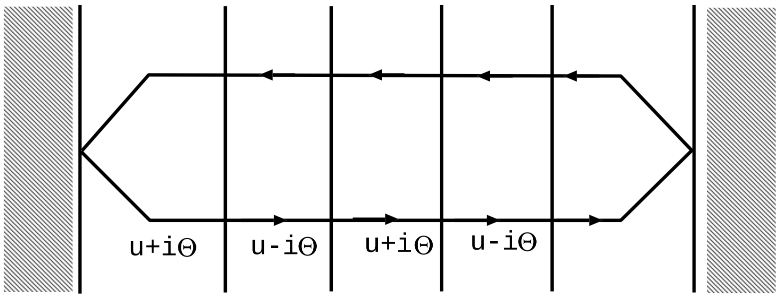
$$\Delta = -\cos\gamma$$

$$p = \sin\gamma\cot\frac{\gamma}{2}(H_+ - 1), \qquad p' = \sin\gamma\cot\frac{\gamma}{2}(H_- - 1)$$

- Consider $0 < p < p_C = 1 + \cos\gamma$ (no boundary string) and $p = p'$.

The 6V model with boundaries

- A vertex model with the 6 possible configuration ($N \times N$ sites)
- The Boltzmann weights: parameterized by u, γ,ρ e.g., $w(+,+,+,+) = \rho \sin(2+u)\gamma$
- an inhonomeneous model, $u \to u \pm i\Theta$



The scaling limit

$$N \to \infty \qquad\qquad \Theta \to \frac{2}{\pi}\log\frac{4N}{mL}$$

$$u \to \frac{2}{\pi}i\theta$$

Correspondence: $m =$ sG Kink mass, $\beta^2 = 8(\pi - \gamma)$.

The diagonalization of T

- The eigenvalue of the inhomogeneous 6V model

$$T_1(x) = B_-(x)T_0(x+i)\frac{Q(x-2i)}{Q(x)} + B_+(x)T_0(x-i)\frac{Q(x+2i)}{Q(x)}$$

where $u = x/i$ and

$$B_\pm(x) = [x \pm iH_+][x \pm iH_-], \quad T_0(x) = [2x][x+\Theta]^N[x-\Theta]^N$$

$$Q(x) = \prod_{j=1}^{N/2}[x-x_j][x+x_j] \quad [x] := \sinh\frac{\gamma}{2}$$

- The energy of the XXZ model

$$E = i\frac{d}{dx}\log T_1(x)|_{x=i}$$

The fusion relations

- Introduce fusion transfer matrices $T_m(x) = \mathrm{T}r_{V_m}\mathcal{T}(x)$ (dim $V_m = m+1$.)
- Functional Relations

$$T_m(x+i)T_m(x-i) = f_m(x) + T_{m+1}(x)T_{m-1}$$
$$f_m(x) = T_0(x+(m+1)i)T_0(x-(m+1)i)$$
$$\times \prod_{j=1}^{m} B_-(x+(m+2-2j)i)B_+(x-(m+2-2j)i)$$

- The normalization (the ground state):
  - $T_m =$ degree N in $[x]$ under the p.b.c
  - $T_m =$ degree $2N+2m$ in the case with boundaries

$T - Y$ transformation

- The $T - Y$ transformation (Klümper-Pearce ('91), Kuniba -Nakanishi-JS('94))

$$Y_j(x) = \frac{T_{j-1}(x)T_{j+1}(x)}{f_j(x)} \quad j = 1, 2, \cdots$$

- The universal $Y-$ system (Al. B. Zamolodchikov)

$$Y_j(x+i)Y_j(x-i) = (1+Y_{j+1}(x))(1+Y_{j-1}(x)), \qquad j = 1, 2, \cdots$$

Truncation of relations

- The peculiarity at $\gamma = \pi/(p+1)$ (BLZ '96)

$$T_{p+1}(x) = T_{p-1}(x)B_-(x-pi)B_+(x+pi)$$
$$+ T_0(x+(p+1)i)\prod_{j=1}^{p} B_-(x+(p+1-2j)i)$$
$$+ T_0(x-(p+1)i)\prod_{j=1}^{p} B_+(x+(p+1-2j)i)$$

or symbolically $T_{p+1}(x) = 2 + T_{p-1}(x)$

- factorization $\gamma = \pi/(p+1)$ (symbolically)

$$T_p(x+i)T_p(x-i) = (1+T_{p-1}(x))^2$$

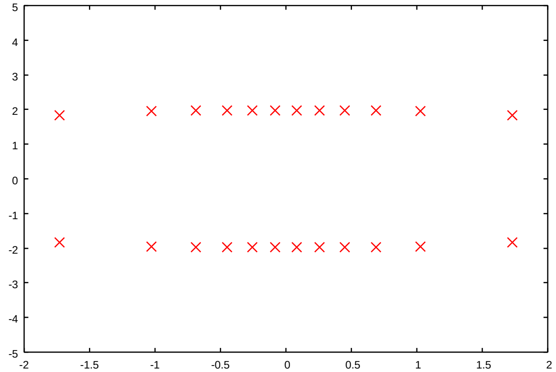
Truncated $Y-$ system

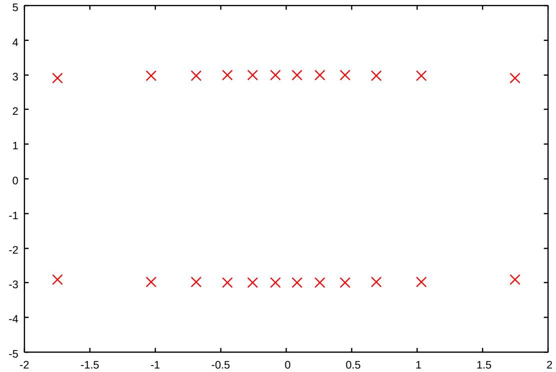
$$
\begin{aligned}
Y_j(x+i)Y_j(x-i) &= (1+Y_{j+1}(x))(1+Y_{j-1}(x)), \quad j = 1, \cdots, p-3 \\
Y_{p-2}(x+i)Y_{p-2}(x-i) &= (1+Y_{p-3}(x))(1+\lambda(x)Y_t(x)) \\
&\times (1+\lambda(\bar{x})Y_t(x)) \\
Y_{t-1}(x+i)Y_{t-1}(x-i) &= (1+Y_{p-2}(x))
\end{aligned}
$$

D_p Dynkin type equations.

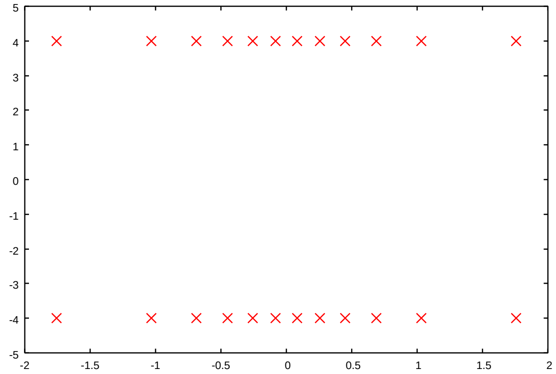
Zeros of T in the repulsive regime

$N = 24, p.b.c., \gamma = \frac{\pi}{4}$





T_1 (left) T_2 (right)



T_3

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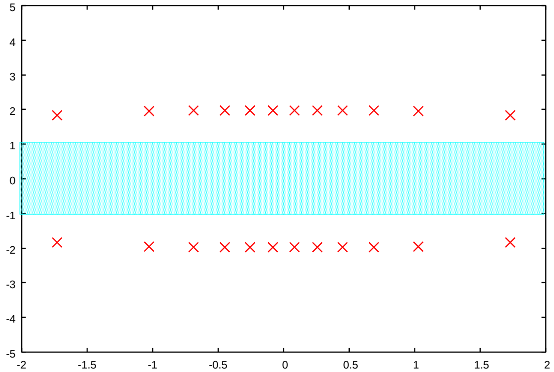

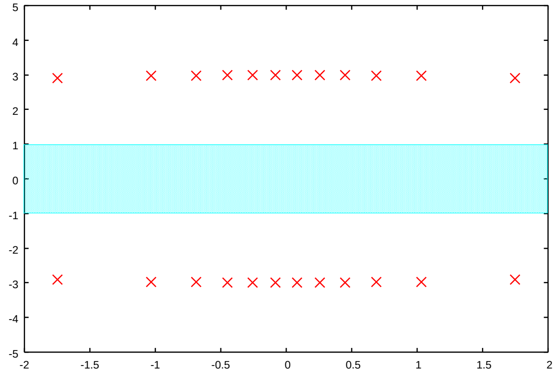

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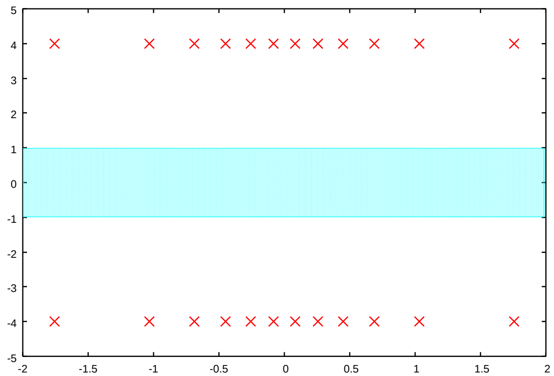


T_3

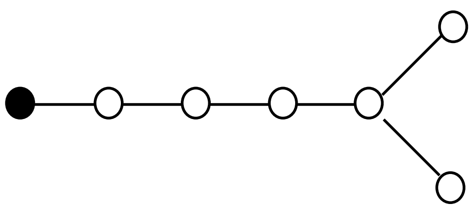
The analyticity and TBA

Analyticity conjecture:

- $Y_j(j = 1, \cdots, p-2, t)$ are analytic and nonzero in the strip $\Im z \in [-1, 1]$ (except for trivial poles)
- $1 + Y_j(j = 1, \cdots, p-2, t)$ are analytic and nonzero in the strip $\Im z \in [-0^+, 0^+]$

1. Truncated Y system
2. Analytic property

$\Longrightarrow$ TBA with a massive node



The energy of the spin chain

Suppose one solves TBA and obtains Y. Then the functional relation

$$T_1(x+i)T_1(x-i) = f_1(1+Y_1(x))$$

and the analyticity determines $T(x)$ in the strip $\Im x \in [-1,1]$. The energy of the spin chain can be evaluated, as

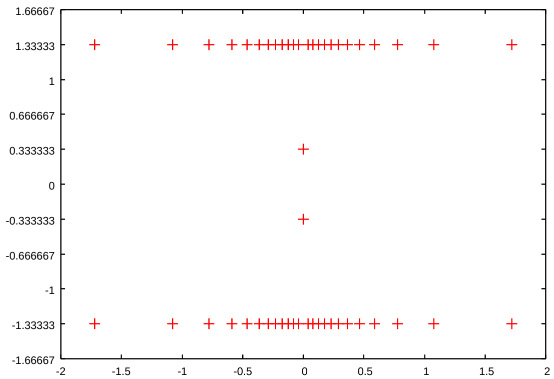
$$E = i\frac{d}{dx}\log T_1(x)|_{x=i}$$

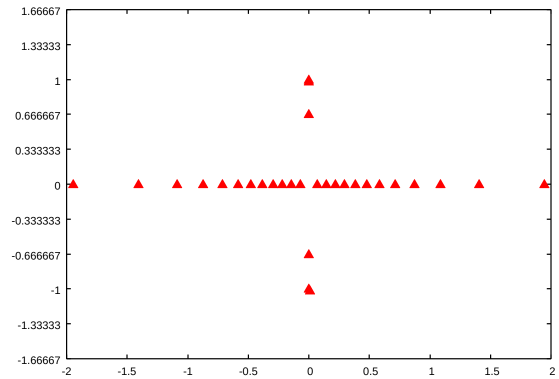
What happens in the attractive regime?

Concentrate on the case $\gamma = 3\pi/4$ with periodicity, $T(x + 8/3i) = T(x)$

Zeros of T in the attractive regime

$N = 12, p = p' = 0.1, \gamma = \frac{3\pi}{4}$





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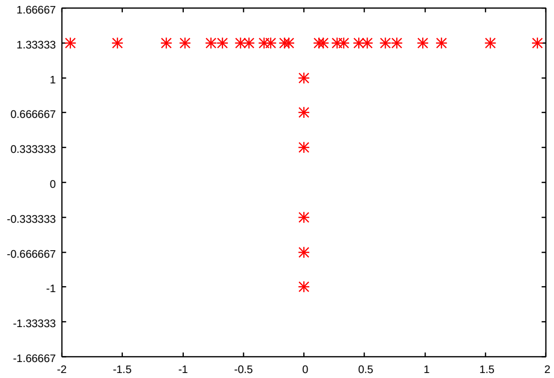

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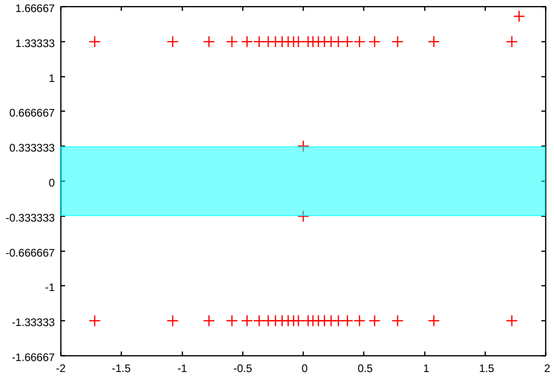

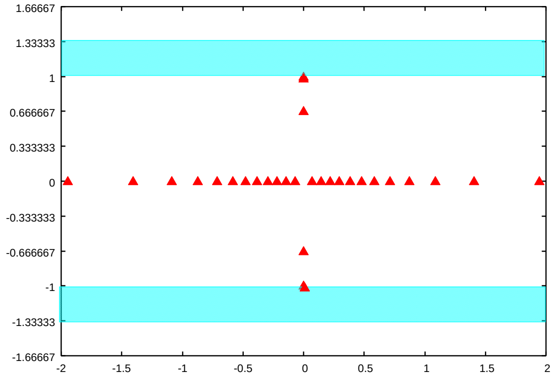

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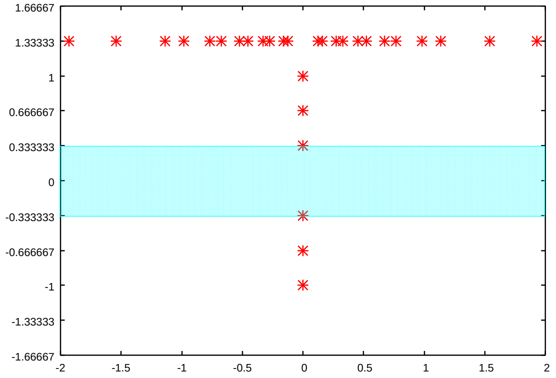

T_3

Relevant T system

- the original T system

$$T_1(x+i)T_1(x-i) = f_1(x) + T_0(x)T_2(x)$$
$$T_2(x+i)T_2(x-i) = f_2(x) + T_1(x)T_3(x)$$
$$T_3(x+i)T_3(x-i) = f_3(x) + T_2(x)T_4(x)$$

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- the shifted T system Let $T^\vee(x) = T^\vee(x + 4/3i)$

$$T_1^\vee(x+i)T_1^\vee(x-i) = f_1^\vee(x) + T_0^\vee(x)T_2^\vee(x)$$
$$T_2^\vee(x+i)T_2^\vee(x-i) = f_2^\vee(x) + T_1^\vee(x)T_3^\vee(x)$$
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- the shifted T system Let $T^\vee(x) = T^\vee(x+4/3i)$

$$T_1(x+\frac{1}{3}i)T_1(x-\frac{1}{3}i) = f_1^\vee(x) + T_0^\vee(x)T_2^\vee(x)$$
$$T_2(x+\frac{1}{3}i)T_2(x-\frac{1}{3}i) = f_2^\vee(x) + T_1^\vee(x)T_3^\vee(x)$$
$$T_3(x+\frac{1}{3}i)T_3(x-\frac{1}{3}i) = f_3^\vee(x) + T_2^\vee(x)T_4^\vee(x)$$

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Relevant Y system

- Define Y functions

$$Y_1(x) = T_0^{\vee}(x) T_2^{\vee}(x) / f_1^{\vee}(x)$$

$$Y_2(x) = T_1(x) T_3(x) / f_2(x)$$

$$Y_t(x) = T_2^{\vee}(x) / T_0(x) B_-(x - i/3) B_+(x + i/3)$$

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- Y system

$$Y_1(x + i/3)Y_1(x - i/3) = (1 + Y_2(x))$$
$$Y_2(x + i/3)Y_2(x - i/3) = (1 + Y_1(x)))$$
$$\times (1 + \lambda(x)Y_t(x))(1 + \lambda'(x)Y_t(x))$$
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- Y system

$$Y_1(x + i/3) Y_1(x - i/3) / Y_2(x) = (1 + (Y_2(x))^{-1})$$
$$Y_2(x + i/3) Y_2(x - i/3) / (Y_1(x) Y_t^2(x)) = (1 + (Y_1(x))^{-1}) \times (\lambda(x) + (Y_t(x))^{-1})(\lambda'(x) + (Y_t(x))^{-1})$$
$$Y_t(x + i/3) Y_t(x - i/3) / Y_2(x) = (1 + (Y_2(x))^{-1})$$

TBA in the attractive regime

- Y system + analyticity assumption
  $\Longrightarrow$ recovers TBA with the correct mass ratio

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- Is the problem solved?

TBA in the attractive regime

- Y system + analyticity assumption
  $\Longrightarrow$ recovers TBA with the correct mass ratio
- Is the problem solved?
- No. $T_1(x)$ known in $\Im x \in [-1/3, 1/3]$
  The evaluation of the energy though

$$E = i \frac{d}{dx} \log T_1(x)|_{x=i}$$

is not yet possible.

A trial?

The analytic continuation of the equation? (i.e., second determination in NLIE.)

- an equation valid in the narrow strip

$$\log T_1(x) = \frac{1}{2\pi}\int K_{\frac{1}{3}}(x-x')\log(1+Y_1(x'))dx'$$

$$K_{\frac{1}{3}}(x) = \frac{3\pi}{2\cosh\frac{3\pi}{2}(x)}$$

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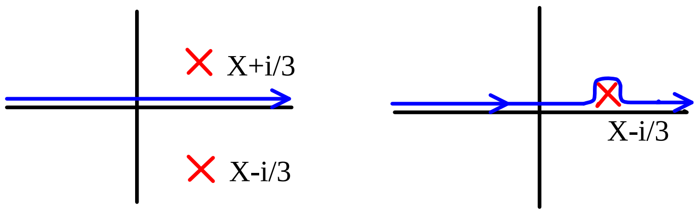
- analytic continuation ($\Im x = 0$ to $\Im x > 1/3$)



Contour deformation

- Deform the contour to the straightline,

$$\log T_1(x+2/3i) = \log(1+Y_1(x+1/3i)) - \frac{1}{2\pi}\int_{-\infty}^{\infty} K_{\frac{1}{3}}(x-x')\log(1+Y_1(x'))dx'$$

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- Using functional relation

$$\log T_1(x+2/3i) = \log T_1(x+2/3i) + \log T_1(x) - \frac{1}{2\pi}\int K_{\frac{1}{3}}(x-x')\log(1+Y_1(x'))dx'$$

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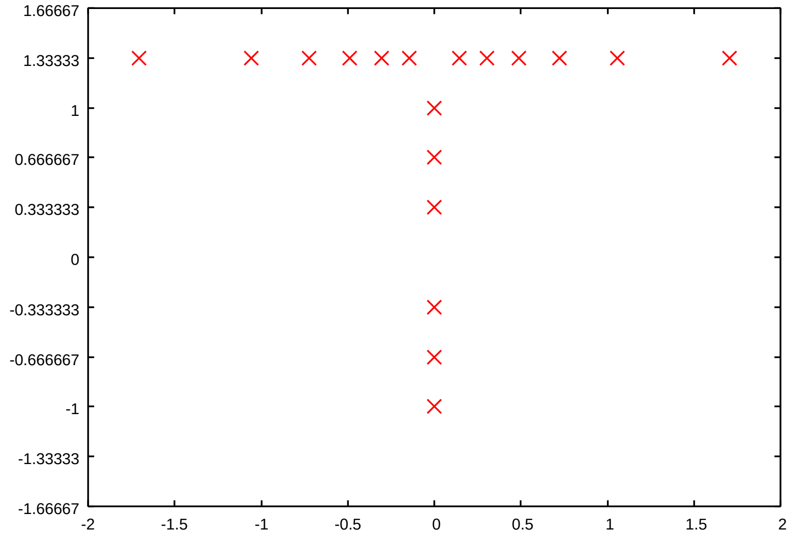
- Come back to the orinal form..nonsense!

$$\log T_1(x) = \frac{1}{2\pi}\int K_{\frac{1}{3}}(x-x')\log(1+Y_1(x'))dx'$$

Zeros of T_3 in the attractive regime

$$N = 12, p = p' = 0, \gamma = \frac{3\pi}{4}$$



Factorized T_3

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  $T_3(x) \propto Q(x - 4/3i)Q(x + 4/3i)$.
- for nonzero boundary fields p, let $\bar{Q}(x)$ be Q for a model with $-p$. then

$$T_3(x) \propto Q(x - 4/3i)\bar{Q}(x + 4/3i)$$

.

The puzzle is solved

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$$E \sim i \frac{d}{dx} \log \frac{Q(x-i)}{Q(x+i)} \Big|_{x=0}$$

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$$E \sim i \frac{d}{dx} \log \frac{Q(x-i)}{Q(x+i)} |_{x=0}$$

- Use the factorized T_3

$$E \sim i \frac{d}{dx} \log \frac{T_3(x+i/3)}{T_3(x-i/3)} |_{x=0}$$

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- THANK YOU