Boundary S Matrices of Massive $\phi_{1,3}$ Perturbed Unitary Minimal Models

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Integrable A_m Models: History

- 1984: Belavin-Polykov-Zamolodchikov
  A_m minimal CFTs

- 1984: Andrews-Baxter-Forrester
  RSOS A_m bulk weights

- 1990: Zamolodchikov; Bernard-LeClair; Reshetikhin-Smirnov
  A_m bulk S matrices

- 1994: Ghoshal-Zamolodchikov
  Boundary S matrices in Integrable QFT
  Ising A_3 boundary S matrices

- 1996: Chim
  Tricritical Ising A_4 boundary S matrices

- 2001: Behrend-Pearce
  RSOS A_m boundary weights

- 2005: Nepomechie-Pearce
  A_m boundary S matrices

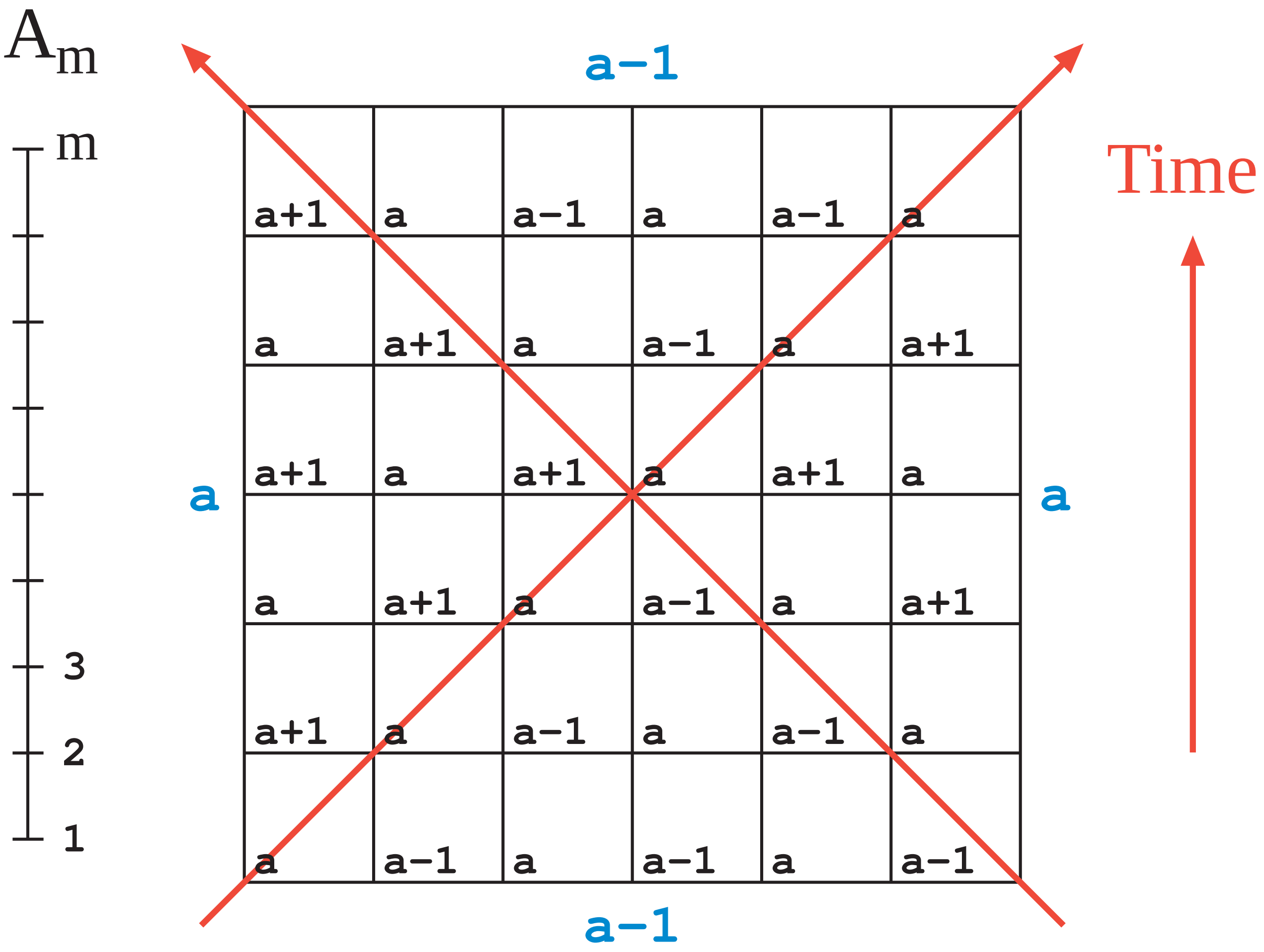
Integrable A_m Scattering Theories

- The integrable A_m scattering theories correspond to the massive continuum scaling limit of the off-critical A_m RSOS models of Andrews-Baxter-Forrester in Regime III.

- These models exhibit $m-1$ coexisting phases (vacua) with alternating sublattice heights a, $a+1$ where $a = 1, 2, \ldots, m-1$.

- The excitation spectrum consists of *massive kinks* (domain walls or particles) that interpolate between neighbouring vacua.

heights $\in A_m$, vacua $\in A_{m-1}$, kinks $\in A_{m-2}$



- This correspondence makes sense at the level of the massive continuum scaling limit — there is no correspondence at the level of lattice configurations.

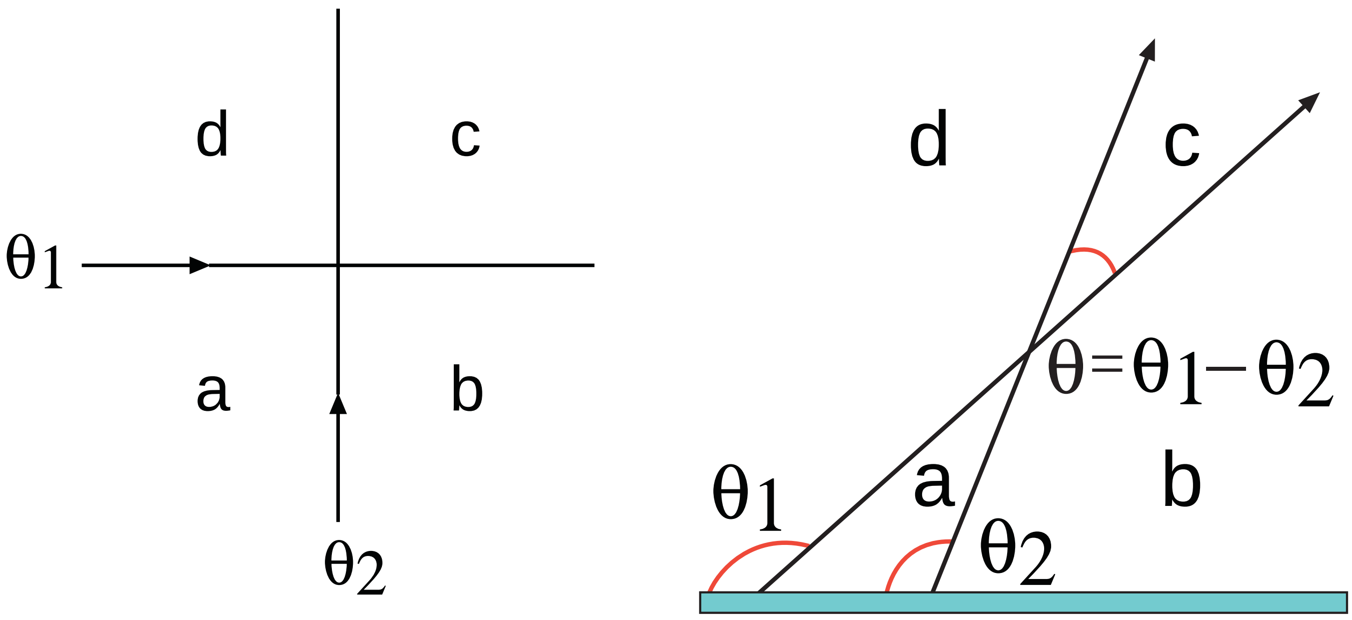
Bulk S Matrices

- A_m bulk S matrices: $(a,b,c,d\in A_{m-1})$

$$
\begin{aligned}
S^{d\ c}_{a\ b}(\theta) &= -\frac{U(\theta)}{i\pi}\left(\frac{[a][c]}{[b][d]}\right)^{-\frac{\theta}{2\pi i}}\overline{S}^{d\ c}_{a\ b}(\theta), \qquad [a]=\frac{\sin(\frac{\pi a}{m})}{\sin(\frac{\pi}{m})}\\
&= \text{physical scattering amplitude}\\
&= S^{d\ a}_{c\ b}(\theta) = S^{b\ c}_{a\ d}(\theta) \qquad \text{(diagonal reflection)}\\
&= S^{\bar d\ \bar c}_{\bar a\ \bar b}(\theta), \qquad \bar a = m-a \qquad \text{(height reversal)}
\end{aligned}
$$

- Reduced matrix elements:

$$
\boxed{\overline{S}^{d\ c}_{a\ b}(\theta) = \sinh(\frac{i\pi-\theta}{m})\delta_{ac} + \left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}}\sinh(\frac{\theta}{m})\delta_{bd}}
$$



- Kink scattering (Zamolodchikov-Fateev algebra):

$$
K_{d,a}(\theta_1)\ K_{a,b}(\theta_2) = \sum_d S^{d\ c}_{a\ b}(\theta_1-\theta_2)\ K_{d,c}(\theta_2)\ K_{c,b}(\theta_1)
$$

- Relativistic energy-momenta in $1+1$ dimensions $(c=1)$:

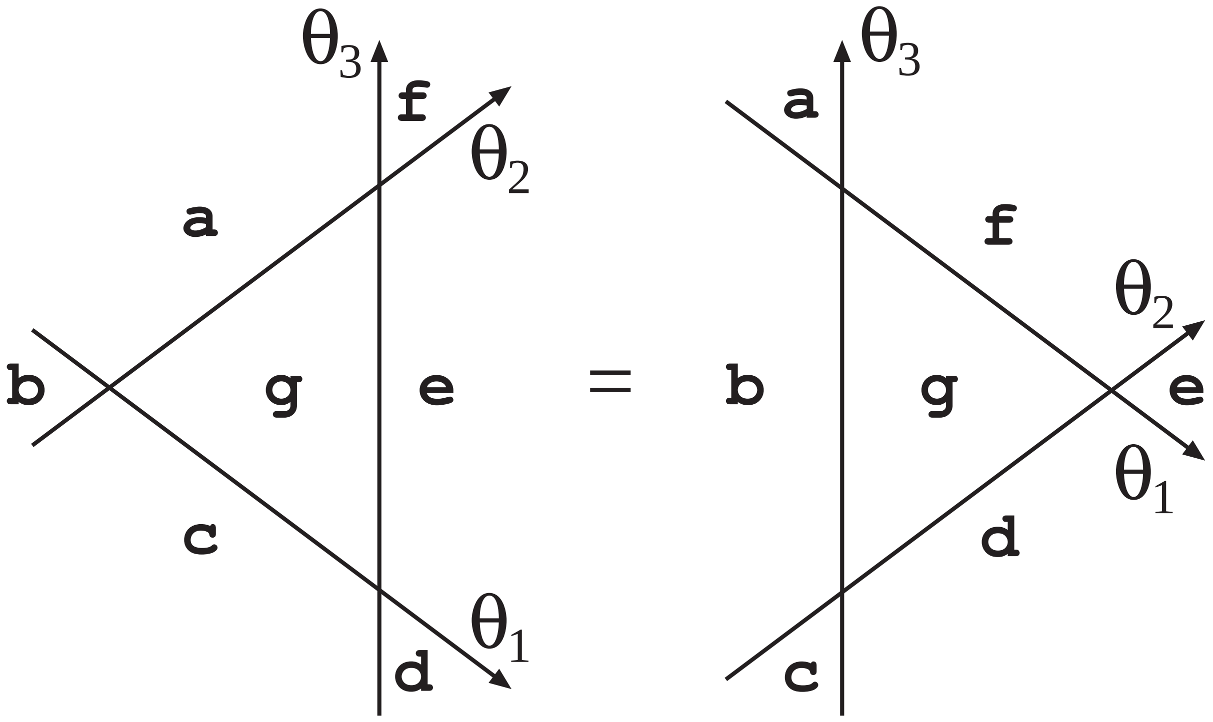
$$
e^2-p^2=M^2, \qquad e=M\cosh\theta, \qquad p=M\sinh\theta
$$

$$
M=\text{kink mass}, \qquad \theta=\text{rapidity}
$$

Yang-Baxter Equation

- Yang-Baxter equation (factorizable S matrices):

$$\sum_g S_{b\ c}^{a\ g}(\theta_1-\theta_2)\ S_{c\ d}^{g\ e}(\theta_1-\theta_3)\ S_{g\ e}^{a\ f}(\theta_2-\theta_3)$$
$$=\sum_g S_{c\ d}^{b\ g}(\theta_2-\theta_3)\ S_{b\ g}^{a\ f}(\theta_1-\theta_3)\ S_{g\ d}^{f\ e}(\theta_1-\theta_2)$$



- RSOS Lattice/QFT solutions:

$$\begin{aligned} W_{a\ b}^{d\ c}(\theta) &= \sin(\lambda-u)\delta_{ac}+\left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}}\sin u\ \delta_{bd} \\ \overline{S}_{a\ b}^{d\ c}(\theta) &= \sinh(\frac{i\pi-\theta}{m})\delta_{ac}+\left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}}\sinh(\frac{\theta}{m})\delta_{bd} \end{aligned}$$

- RSOS Lattice/QFT correspondence:

$$u\mapsto\frac{\theta}{im},\qquad \lambda=\frac{\pi}{m}$$

- Moral:

Massive A_m QFT = critical A_{m-1} RSOS lattice
with "physical" normalization and gauge!

Crossing and Unitarity

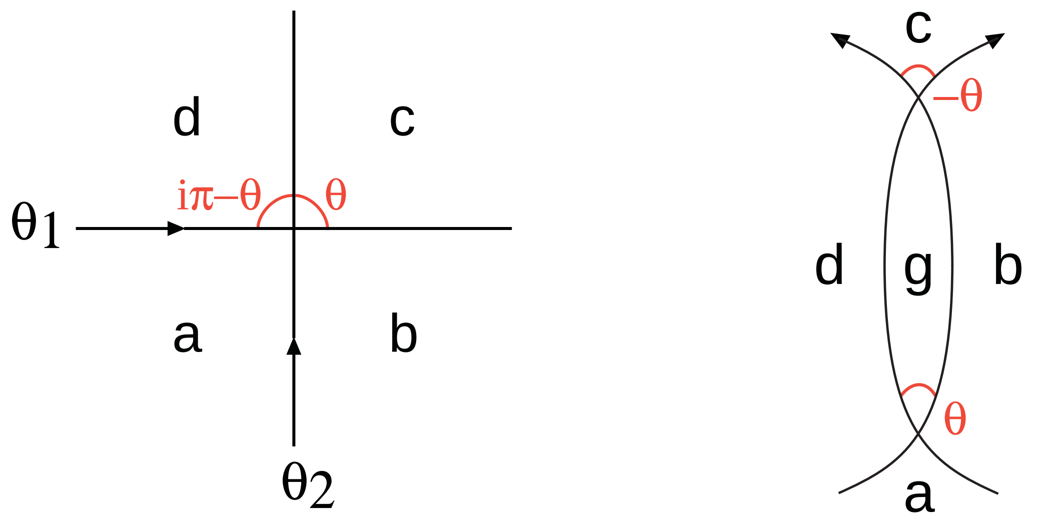
- Bulk S matrices:

$$S^{d}_{a}{}^{c}_{b}(\theta) \;=\; -\frac{U(\theta)}{i\pi}\left(\frac{[b][d]}{[a][c]}\right)^{-\frac{\theta}{2\pi i}} \overline{S}^{d}_{a}{}^{c}_{b}(\theta)$$

- Crossing relation fixes gauge:

$$S^{d}_{a}{}^{c}_{b}(\theta) = S^{a}_{d}{}^{b}_{c}(i\pi-\theta), \qquad U(i\pi-\theta) = U(\theta)$$

$$W^{d}_{a}{}^{c}_{b}(u) = \left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}} W^{a}_{d}{}^{b}_{c}(\lambda-u)$$



- Unitarity fixes normalization:

$$\sum_g S^{d}_{a}{}^{g}_{b}(\theta)\; S^{d}_{g}{}^{c}_{b}(-\theta) = \delta_{ac}$$

$$\sum_g W^{d}_{a}{}^{g}_{b}(u)\; W^{d}_{g}{}^{c}_{b}(-u) = \sin(u+\lambda)\sin(u-\lambda)\delta_{ac}$$

- The scalar factor $U(\theta) = U(i\pi-\theta)$ satisfies

$$U(\theta)\; U(-\theta) = \frac{\pi^2}{\sinh(\frac{1}{m}(\theta+i\pi))\sinh(\frac{1}{m}(\theta-i\pi))}$$

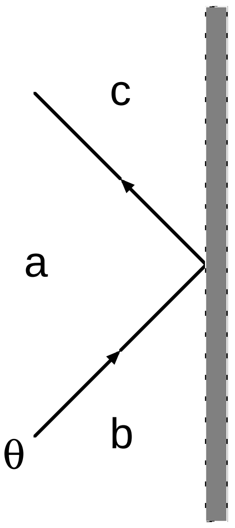
An integral representation for this function in $0 \le \operatorname{Im}\theta < \pi$ is

$$U(\theta) = \frac{-i\pi}{\sinh(\frac{1}{m}(\theta-i\pi))}\exp\left(i\int_0^\infty \frac{dt}{t}\,\frac{\sin(\frac{\theta t}{\pi})\sinh(\frac{(m-1)t}{2})}{\sinh(\frac{mt}{2})\cosh(\frac{t}{2})}\right)$$

Boundary S Matrices

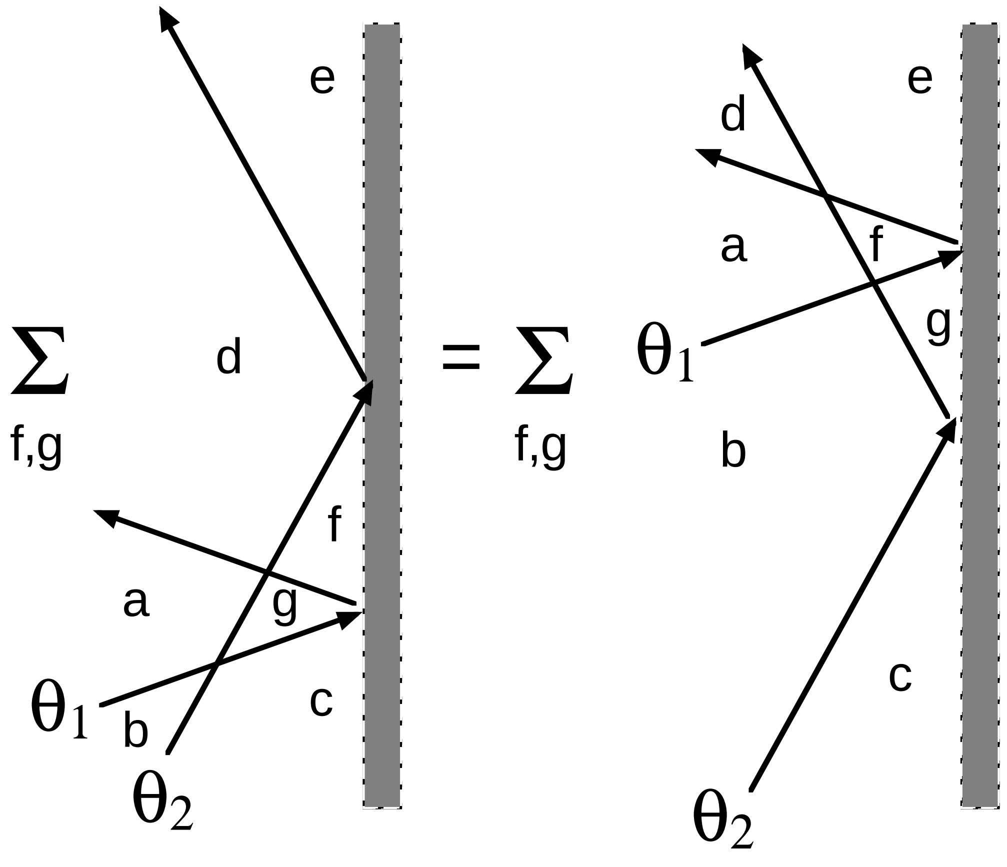
- Boundary S matrices:

$$K_{a,b}(\theta)\;B_b=\sum_c R^{(r,s)}\,{}_{a}{}^{c}_{b}(\theta\,,\xi)\;K_{a,c}(-\theta)\;B_c$$



- Boundary Yang-Baxter equation:

$$\sum_{f,g} S^{a}_{b}{}^{g}_{c}(\theta_1-\theta_2)R^{(r,s)}\,{}_{g}{}^{f}_{c}(\theta_1\,,\xi)S^{a}_{g}{}^{d}_{f}(\theta_1+\theta_2)R^{(r,s)}\,{}_{d}{}^{e}_{f}(\theta_2\,,\xi)$$
$$=\;\sum_{f,g} R^{(r,s)}\,{}_{b}{}^{g}_{c}(\theta_2\,,\xi)S^{a}_{b}{}^{f}_{g}(\theta_1+\theta_2)R^{(r,s)}\,{}_{f}{}^{e}_{g}(\theta_1\,,\xi)S^{a}_{f}{}^{d}_{e}(\theta_1-\theta_2)$$



Elementary Solutions

- The elementary solutions are the building blocks to build the boundary S matrices with the required properties.

- A_{m-1} (r,s) Behrend-Pearce RSOS lattice solutions of BYBE: (with $r,s=1,2,\ldots,m-1$ and $u\mapsto -i\theta/m\,,\ \xi\mapsto\xi/m$ and $\mu=\pm1$)

$$\overline{B}^{(r,s)}\,{}_{c}\,{}^{c\pm1}_{c\pm1}(\theta,\xi,\mu) =$$

$$\frac{1}{[r]\sqrt{[c][c\pm1]}}\Big[[(r\mp c+s)/2][(c\pm s\mp r)/2]s(i\xi+\theta)\ \ s(i(r\pi+\xi)-\theta)$$

$$+\,[(r\pm c+s)/2][(c\mp s\pm r)/2]s(i\xi-\theta)\ \ s(i(r\pi+\xi)+\theta)\Big]F^{r}_{c\pm1,s}$$

$$\overline{B}^{(r,s)}\,{}_{c}\,{}^{c\mp1}_{c\pm1}(\theta,\xi,\mu) =$$

$$\mu\, s(2\theta)\frac{\sqrt{[(r-c+s)/2][(r+c-s)/2][(c+s-r)/2][(c+s+r)/2]}}{\sqrt[4]{[c-1][c+1]}\sqrt{[c]}}$$

$$\times\ F^{r}_{c+1,s}F^{r}_{c-1,s}$$

where

$$s(x)=[-ix]=\frac{\sinh(\frac{x}{m})}{i\sin(\frac{\pi}{m})}$$

- The matrices F^r are the fused adjacency matrices of A_{m-1}. For example, for A_4:

$$F^1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad F^2=\begin{pmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{pmatrix}$$

$$F^3=\begin{pmatrix}0&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&0\end{pmatrix},\quad F^4=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

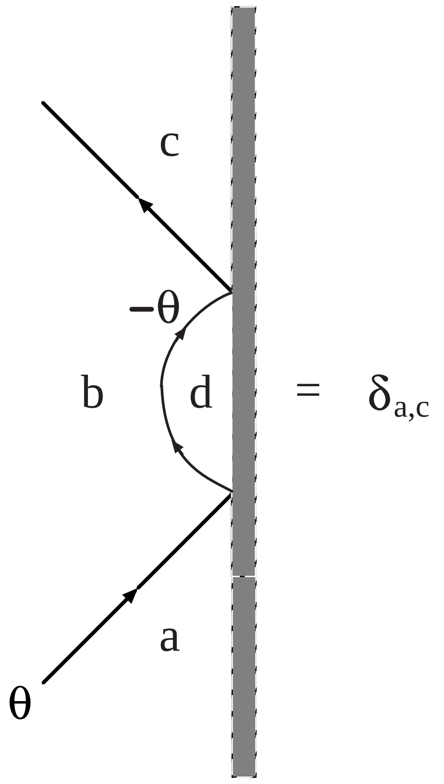
Boundary Unitarity and Crossing

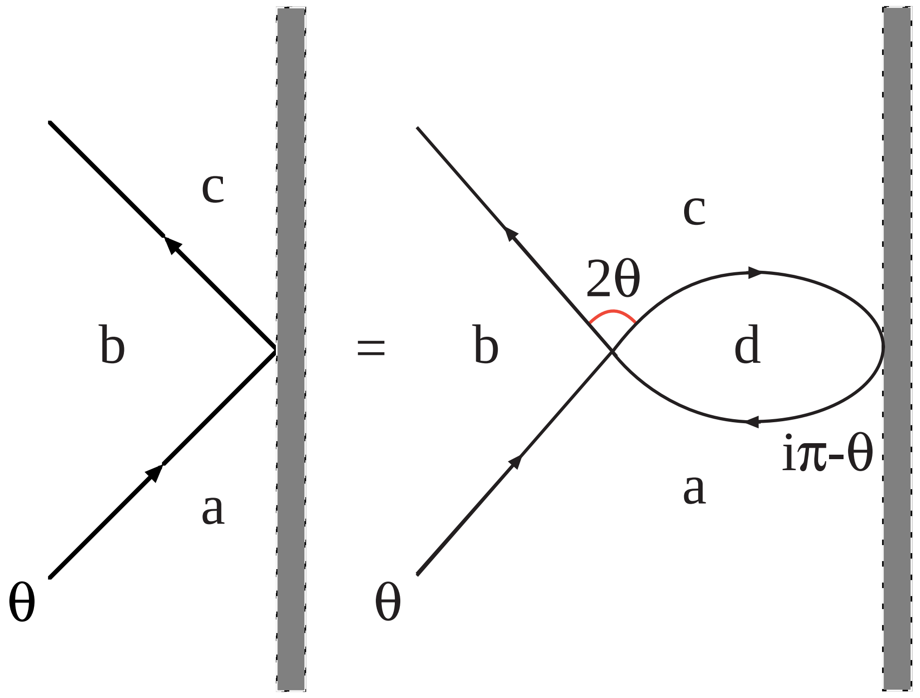
- Boundary unitarity:

$$\sum_d R^{(r,s)}{}_b{}^{d}_{a}(\theta\,,\xi)\;R^{(r,s)}{}_b{}^{c}_{d}(-\theta\,,\xi)=\delta_{a\,,c}$$

- Boundary crossing:

$$R^{(r,s)}{}_b{}^{c}_{a}(\theta\,,\xi)=\sum_d S^{b\;\,c}_{a\;\,d}(2\theta)\;R^{(r,s)}{}_d{}^{c}_{a}(i\pi-\theta\,,\xi)$$





- These fix the normalization and gauge so that

$$\boxed{B^{(r,s)}{}_a{}^{c}_{b}(\theta,\xi,\mu)=V^{(r,s)}(\theta,\xi,\mu)\left(\frac{[b]\,[c]}{[a]^2}\right)^{\frac{i\pi/2-\theta}{2\pi i}}\overline{B}^{(r,s)}{}_a{}^{c}_{b}(\theta,\xi,\mu)}$$

satisfies BYBE, unitarity and crossing for a suitable scalar factor $V^{(r,s)}(\theta,\xi,\mu)$.

Scalar Factor

- From unitarity and crossing the scalar factor must satisfy

$$\begin{aligned} & V^{(r,s)}(\theta\,,\xi,\mu)\,V^{(r,s)}(-\theta,\xi,\mu) \\ \times\quad & s(i(r\pi+\xi)+\theta)\,s(i(r\pi+\xi)-\theta)\,s(i\xi+\theta)\,s(i\xi-\theta)=1 \end{aligned}$$

$$-V^{(r,s)}(\frac{i\pi}{2}+\theta\,,\xi,\mu)\ s(2\theta+i\pi)\ \frac{U(2\theta)}{\pi}\sin(\frac{\pi}{m})=V^{(r,s)}(\frac{i\pi}{2}-\theta,\xi,\mu)$$

- The solution is (lattice then $u\mapsto\frac{\theta}{im}$, $\xi\mapsto\frac{\xi}{m}$)

$$V^{(r,s)}(\theta,\xi,\mu)=\frac{P_0(\pi i-\theta)V^{(r,s)}_{CDD}(\theta,\xi)}{2\cos(\pi/m)s(i(r\pi+\xi))s(i\xi)V_0(\theta)V_r(\theta,\xi)}$$

$$V_0(\theta)\;=\;\exp\Big(2\int_{-\infty}^{\infty}\frac{\sinh\frac{(m-3)\pi t}{2}\sinh\frac{\pi t}{2}\sinh(i\theta t)\sinh(\pi+i\theta)t}{t\,\sinh\frac{m\pi t}{2}\cosh\pi t}\,dt\Big)$$

$$V_r(\theta,\xi)\;=\;\exp(-2\mathcal{I}),\qquad -\frac{\pi}{2}<\mathrm{Re}\,\xi<\pi$$

$$\mathcal{I}=\int_{-\infty}^{\infty}\frac{\cosh\left((m-r-2)\pi+2\xi\right)t\cosh r\pi t\sinh(i\theta t)\sinh(\pi+i\theta)t}{t\,\sinh m\pi t\cosh\pi t}\,dt$$

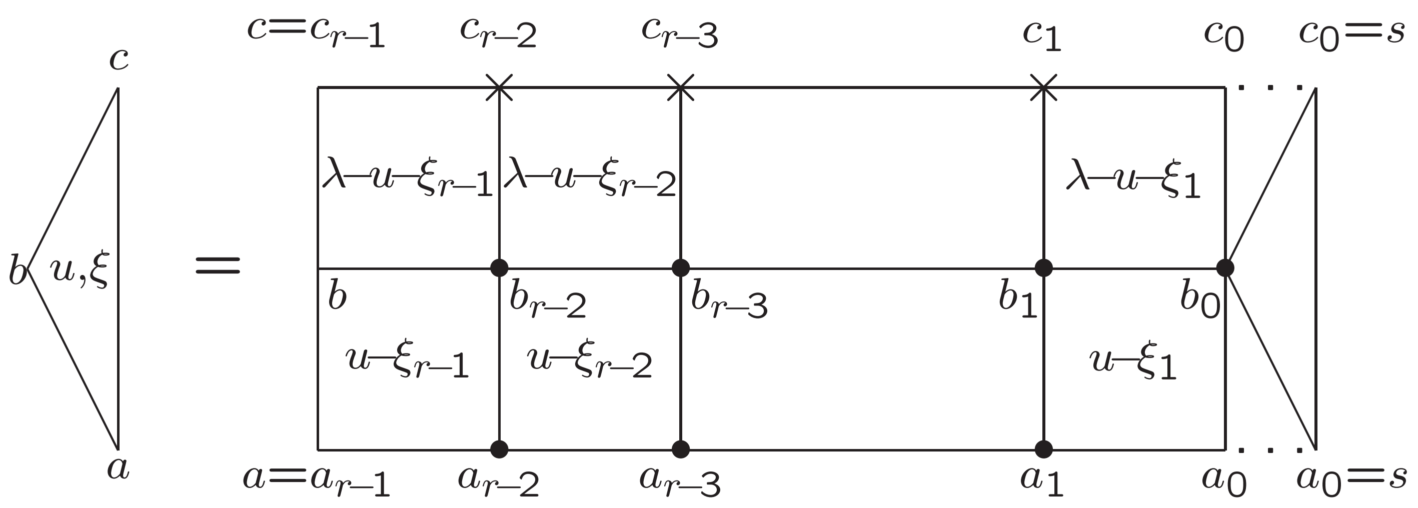
$$P_0(\theta)=\exp\Big(-2\int_0^{\infty}\frac{\sinh\frac{(m-3)\pi t}{2}\sinh\frac{\pi t}{2}\sinh(2i\theta t)}{t\,\sinh\frac{m\pi t}{2}\sinh 2\pi t}\,dt\Big)$$

$$V^{(r,s)}_{CDD}(\theta,\xi)=i\tanh(\frac{i\pi}{4}-\frac{\theta}{2})\frac{\sin\xi-i\sinh\theta}{\sin\xi+i\sinh\theta}$$

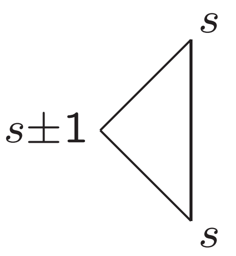
- Explicit elementary solutions exist for m odd.
- Although the amplitudes $B^{(r,s)}\,{}_{a}{}^{c}_{b}(\theta,\xi,\mu)$ satisfy BYBE, unitarity and crossing, they are not the required boundary S matrices since they do not satisfy the boundary bootstrap.

Behrend-Pearce Construction

- (r,s) Solution: $(\xi_j = \xi + j\lambda =$ column inhomogeneities$)$



- The (r,s) solutions are constructed from the $(1,s)$ solutions by fusing with $r-1$ columns.



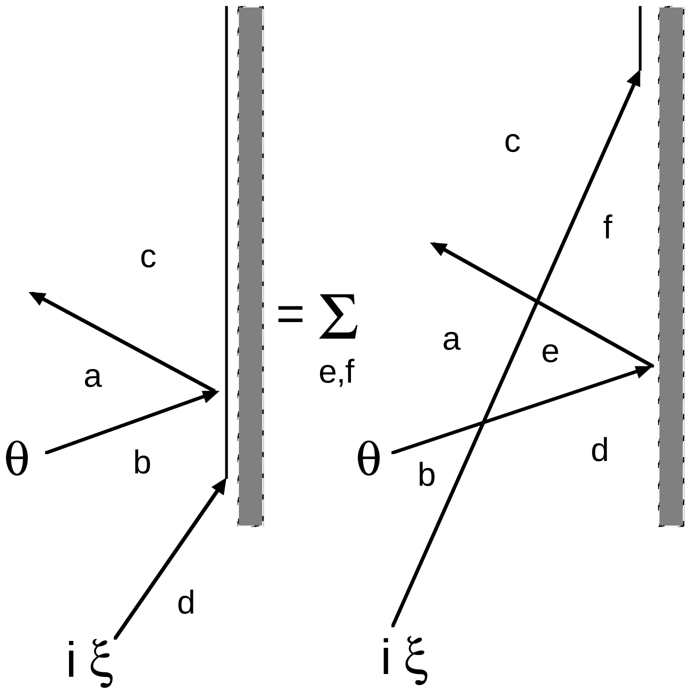
$$= \sqrt{\frac{[s\pm 1]}{[s]}}$$

- Viewing this recursively (with $u \mapsto \frac{\theta}{im}$, $\xi \mapsto \frac{\xi}{m}$) as a relation between the (r,s) and $(r+1,s)$ solutions yields the bootstrap equation.

Boundary Bootstrap

- If a kink with rapidity $\theta = i\xi$ can fuse to the boundary to form a "*boundary bound state*", this implies the boundary bootstrap:

$$g_{d,b}\, R^{(r,s)}\, a\, {}^{c}_{b}(\theta,\xi) = \sum_{e,f} g_{f,c}\, S^{a}_{b}{}^{e}_{d}(\theta - i\xi)\, R^{(r,s)}\, e\, {}^{f}_{d}(\theta,\xi)\, S^{a}_{e}{}^{c}_{f}(\theta + i\xi)$$



- Instead, for $r, s = 1, 2, \ldots, m-1$, it follows from the Behrend-Pearce fusion construction that

$$\frac{s(\theta - i(\xi+\pi))\, s(\theta + i\xi)}{V^{(r+1,s)}(\theta, \xi-\pi, -\mu)}\, g_{d,b}\; B^{(r+1,s)}\, a\, {}^{c}_{b}(\theta, \xi-\pi, -\mu)$$
$$= \frac{\pi^2}{\sin^2(\frac{\pi}{m}) U(\theta - i\xi) U(\theta + i\xi) V^{(r,s)}(\theta,\xi,\mu)}$$
$$\times \sum_{e,f} g_{f,c}\, S^{a}_{b}{}^{e}_{d}(\theta - i\xi) B^{(r,s)}\, e\, {}^{f}_{d}(\theta,\xi,\mu) S^{a}_{e}{}^{c}_{f}(\theta + i\xi)$$

This relates the amplitudes $B^{(r,s)}$ to $B^{(r+1,s)}$.

- The particle-boundary coupling constants are given by

$$g_{a,b} = \frac{1}{g_a\, g_b}\left[\frac{s(\pi i b)}{s(\pi i a)}\right]^{\xi/2\pi}$$

where the factors $g_a = [a]^{p_a/2}$, for suitable $p_a \in \mathbb{Z}$, are related to the lattice fusion vectors of Behrend and Pearce.

Particle Boundary Couplings

- The particle-boundary coupling constants are given by

$$g_{a,b} = \frac{1}{g_a\, g_b} \left[\frac{s(\pi i b)}{s(\pi i a)} \right]^{\xi/2\pi}$$

where $g_a = [a]^{p_a/2}$.

- For $m = 3, 4$: $p_a = 0$ and $g_a = 1$

- For $m = 5$, $p = (p_1, p_2, p_3, p_4)$:

s					
4	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
3	(0,0,0,0)	(0,1,2,0)	(0,0,-1,0)	(0,0,0,0)	
2	(0,0,0,0)	(0,2,1,0)	(0,-1,0,0)	(0,0,0,0)	
1	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
	1	2	3	4	r

- For $m = 6$, $p = (p_1, p_2, p_3, p_4, p_5)$:

s						
5	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	
4	(0,0,0,0,0)	(0,-2,1,0,0)	(0,2,2,0,0)	(0,0,-1,0,0)	(0,0,0,0,0)	
3	(0,0,0,0,0)	(0,0,1,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	
2	(0,0,0,0,0)	(0,2,1,0,0)	(0,-2,2,0,0)	(0,0,-1,0,0)	(0,0,0,0,0)	
1	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	
	1	2	3	4	5	r

Paired Solutions

- For $r = 1, 2, \ldots, m-1$; $s = 1, 2, \ldots, m$, we define paired solutions by the direct sum:

$$\mathcal{R}^{(r,s)}\, a\,{}^{c}_{b}(\theta,\xi,\mu) = B^{(s-1,r)}\, a\,{}^{c}_{b}(\theta,\xi,\mu) \oplus B^{(m-s,m-r)}\, a\,{}^{c}_{b}(\theta,\pi-\xi,\mu)$$

where for $r, s = 1, 2, \ldots, m-1$

$$-\overline{B}^{(s,r)}\, a\,{}^{c}_{b}(\theta,\xi-\pi,-\mu) = \overline{B}^{(m-s,m-r)}\, a\,{}^{c}_{b}(\theta,\pi-\xi,\mu)$$

- Since the boundary spins b, c in the two elementary solutions are of opposite parity (even/odd), these solutions do not mix and automatically satisfy BYBE, unitarity and crossing.

- Moreover, due to the Behrend-Pearce construction, these paired solutions satisfy the boundary bootstrap.

- The paired solutions also satisfy the A_m Kac table symmetry and A_{m-1} height-reversal symmetries:

$$\mathcal{R}^{(r,s)}\, a\,{}^{c}_{b}(\theta,\xi,\mu) = \mathcal{R}^{(r',s')}\, a\,{}^{c}_{b}(\theta,\pi-\xi,\mu)$$

$$\mathcal{R}^{(r,s)}\, a\,{}^{c}_{b}(\theta,\xi,\mu) = \mathcal{R}^{(r',s)}\, \bar{a}\,{}^{\bar{c}}_{\bar{b}}(\theta,\xi,\mu) = \mathcal{R}^{(r,s')}\, \bar{a}\,{}^{\bar{c}}_{\bar{b}}(\theta,\pi-\xi,\mu)$$

where $r' = m - r$, $s' = m + 1 - s$ and $\bar{a} = m - a$.

- For $m = 3, 4$ the paired solutions agree with the boundary S matrices of Ghoshal-Zamolodchikov and Chim (corrected by Miwa-Weston 1997).

Boundary Subsets

- For $r = 1, 2, \ldots, m-1$; $\quad s = 1, 2, \ldots, m$, the set of allowed vacua on the boundary is given by the disjoint union

$$\boxed{\mathcal{U}_{(r,s)} = \mathcal{V}_{(s-1,r)} \cup \mathcal{V}_{(s,r)}}$$

where the *elementary boundary subsets* are

$$\mathcal{V}_{(r,s)} = \{b \in A_{m-1} \,|\, F^r_{bs} > 0\} = \{b \in A_m \,|\, \tilde{F}^r_{bs} \tilde{F}^{r+1}_{b+1\,s} > 0\} \subseteq A_{m-1}$$

with

$$\text{parity}(b) = \text{parity}(r+s+1)$$

- The matrices F^r, $\tilde{F}^r$ are the fused adjacency matrices of A_{m-1}, A_m respectively. For example, for A_4:

$$F^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$F^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- The sets $\mathcal{U}_{(r,s)}$ of allowed vacua for the TIM $(m = 4)$ are:

s				
4	$\{3\}$	$\{2\}$	$\{1\}$	
3	$\{2,3\}$	$\{1,2,3\}$	$\{1,2\}$	
2	$\{1,2\}$	$\{1,2,3\}$	$\{2,3\}$	
1	$\{1\}$	$\{2\}$	$\{3\}$	
	1	2	3	r

Pole Structure and Bootstrap

- The structure of the poles is consistent with the physical interpretation of the boundary bootstrap:

pole in physical strip ⇒ boundary bound state

If $0 < \mathrm{Re}\,\xi < \frac{\pi}{2}$ then $i\xi \in$ physical strip

$$\begin{aligned} \mathcal{V}_{(s-1,r)} &= \{\text{stable boundary states}\} \\ \mathcal{V}_{(s,r)} &= \{\text{(excited) boundary bound states}\} \end{aligned}$$

- $\theta = i\xi$ is a CDD pole of $B^{(s-1,r)}\,{}_{a}{}^{c}_{b}(\theta,\xi,\mu)$ if $a \in \mathcal{V}_{(s,r)}$

- $\theta = i\xi$ is a simple zero of $\overline{B}^{(s-1,r)}\,{}_{a}{}^{c}_{b}(\theta,\xi,\mu)$ if $a \notin \mathcal{V}_{(s,r)}$ which cancels CDD pole

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If $\frac{\pi}{2} < \mathrm{Re}\,\xi < \pi$ then $i(\pi-\xi) \in$ physical strip

$$\begin{aligned} \mathcal{V}_{(s-1,r)} &= \{\text{(excited) boundary bound states}\} \\ \mathcal{V}_{(s,r)} &= \{\text{stable boundary states}\} \end{aligned}$$

- $\theta = i(\pi-\xi)$ is a CDD pole of $B^{(s,r)}\,{}_{a}{}^{c}_{b}(\theta,\xi,\mu)$ if $a \in \mathcal{V}_{(s-1,r)}$

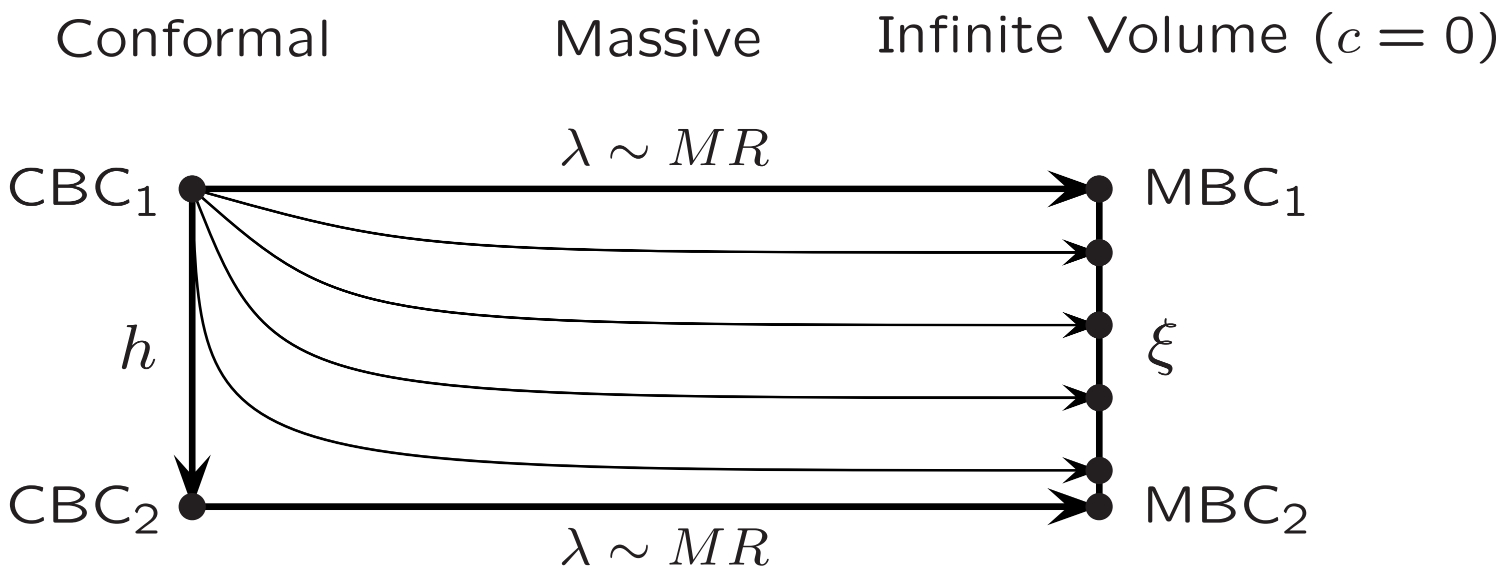
- $\theta = i(\pi-\xi)$ is a simple zero of $\overline{B}^{(s,r)}\,{}_{a}{}^{c}_{b}(\theta,\xi,\mu)$ if $a \notin \mathcal{V}_{(s-1,r)}$ which cancels CDD pole

Boundary Flows

- The action is:

$$A = A_{A_m+(r,s)} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \ \Phi_{1,3}(x\,,y) \pm h \int_{-\infty}^{\infty} dy \ \phi_{1,3}(y)$$

- Two-parameter integrable perturbation of the A_m models $=$ one-parameter family of massive RG flows:



The bulk RG flow is induced by $MR \sim T - T_c$. The "massive boundary flow", which is not an RG flow, is induced by Im $\xi \to \pm\infty$ with h real:

$+$ perturbation: $\quad \xi = \frac{\pi}{2} \mapsto \pi \mapsto \frac{3\pi}{2} \mapsto \frac{3\pi}{2} + i\infty$

$-$ perturbation: $\quad \xi = \frac{\pi}{2} \mapsto 0 \mapsto -\frac{\pi}{2} \mapsto -\frac{\pi}{2} + i\infty$

- The conformal boundary RG flows are given by the Fredenhagen and Schomerus rule:

$$(r,s) \ = \ (r,1)\times(1,s) \ \mapsto \begin{cases} (r,1)\times(s,1) & + \text{ perturbation} \\ (r,1)\times(s-1,1) & - \text{ perturbation} \end{cases}$$

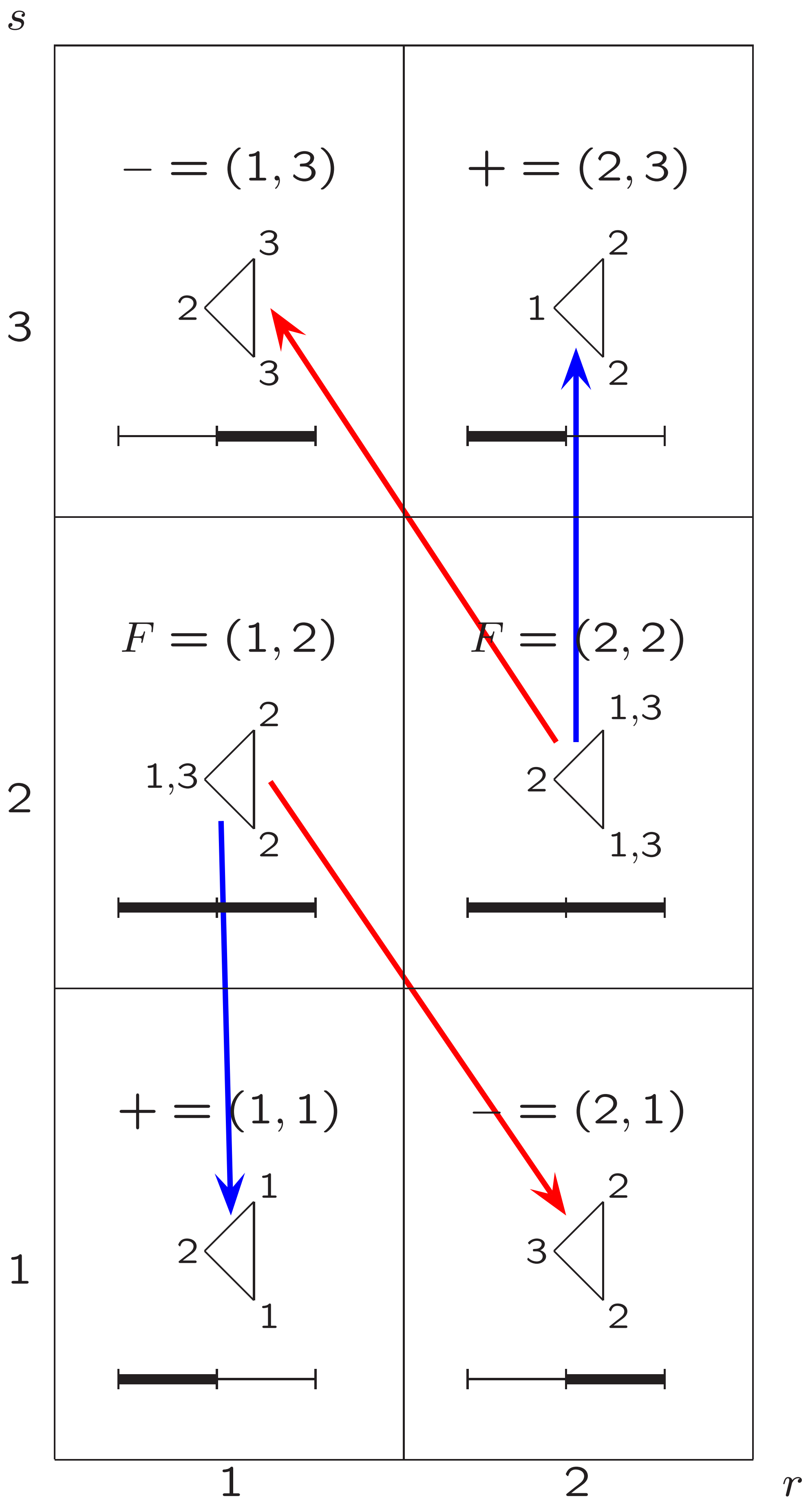
The products are taken in the A_m fusion algebra. The $(r,1)$ CBCs are stable.

- The "massive boundary flows", described by the boundary S matrices, are compatible with the conformal boundary RG flows:

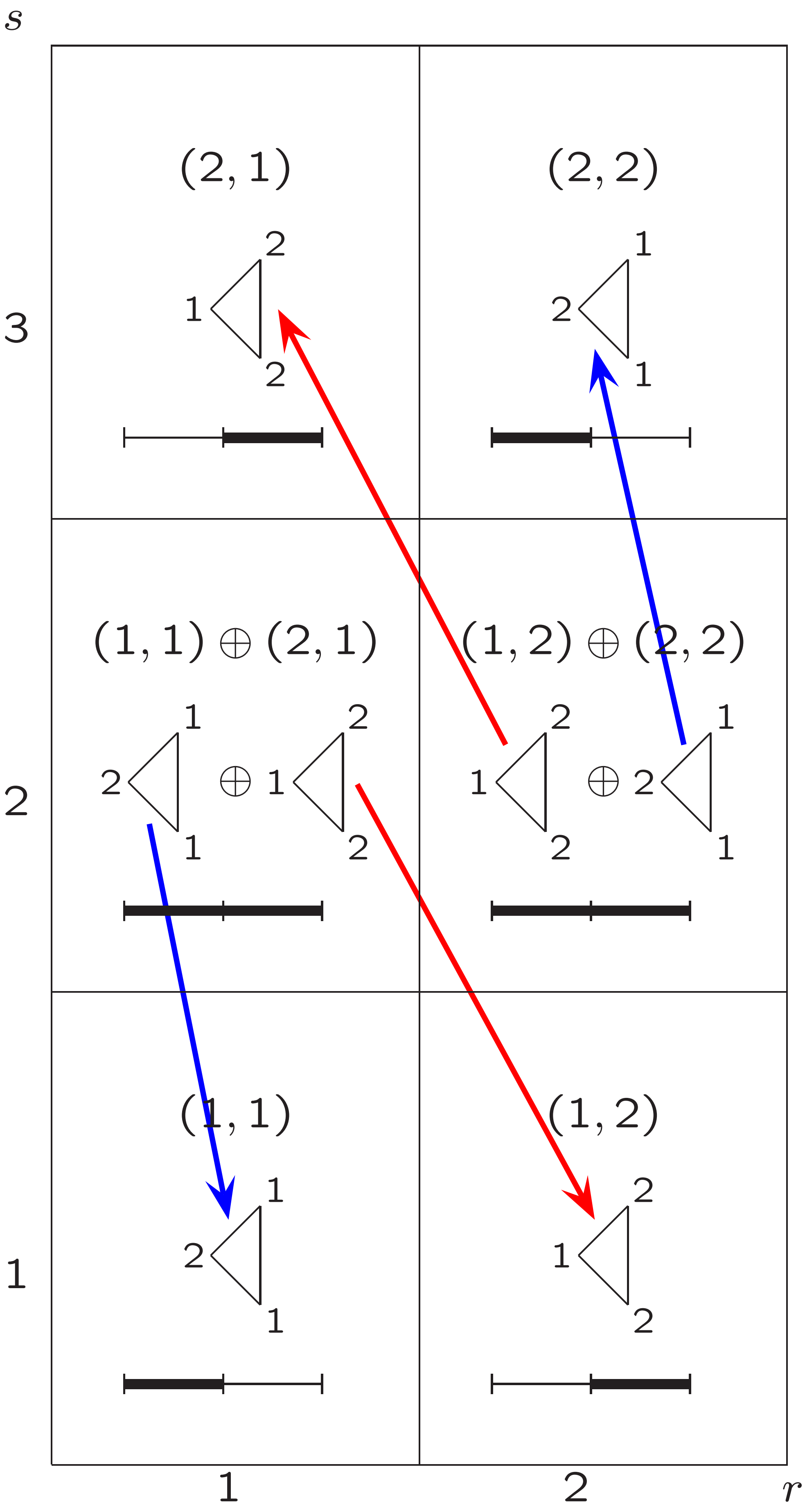
(i) the vacua that live on the boundary coincide

(ii) the patterns of flows are identical

Conformal Ising Model



Massive Ising Model



Ising Boundary S Matrices

$$\boxed{\sin\xi = 1-\frac{h^2}{2M},\qquad h\leftrightarrow -h\quad\Leftrightarrow\quad \xi\leftrightarrow\pi-\xi}$$

$(r,s)=(1,2)$ Interpolating Flow F $\to\pm$:

$$\begin{aligned}\mathcal{R}^{(1,2)}\,a\,{}^{c}_{b}(\theta,\xi,1) &= B^{(1,1)}\,a\,{}^{c}_{b}(\theta,\xi,1)\oplus -B^{(2,1)}\,a\,{}^{c}_{b}(\theta,\xi-\pi,-1)\\ &= B^{(1,1)}\,2\,{}^{1}_{1}(\theta,\xi,1)\oplus B^{(1,2)}\,1\,{}^{2}_{2}(\theta,\pi-\xi,1)\end{aligned}$$

$$B^{(1,1)}\,2\,{}^{1}_{1}(\theta,\xi,\mu)=B^{(1,2)}\,1\,{}^{2}_{2}(\theta,\pi-\xi,\mu)=v_{CDD}(\theta,\xi)$$

$$v_{CDD}(\theta,\xi)=i\tanh\left(\frac{\pi i}{4}-\frac{\theta}{2}\right)\frac{\sin\xi-i\sinh\theta}{\sin\xi+i\sinh\theta}$$

$$\lim_{\mathrm{Im}\,\xi\to\pm\infty} v_{CDD}(\theta,\xi)=1$$

Free F $(\xi=\pi/2)$:

$$B^{(1,1)}\,2\,{}^{1}_{1}(\theta,\xi,1)\Big|_{\xi=\frac{\pi}{2}}=B^{(1,2)}\,1\,{}^{2}_{2}(\theta,\pi-\xi,1)\Big|_{\xi=\frac{\pi}{2}}=-i\coth\left(\frac{\pi i}{4}-\frac{\theta}{2}\right)$$

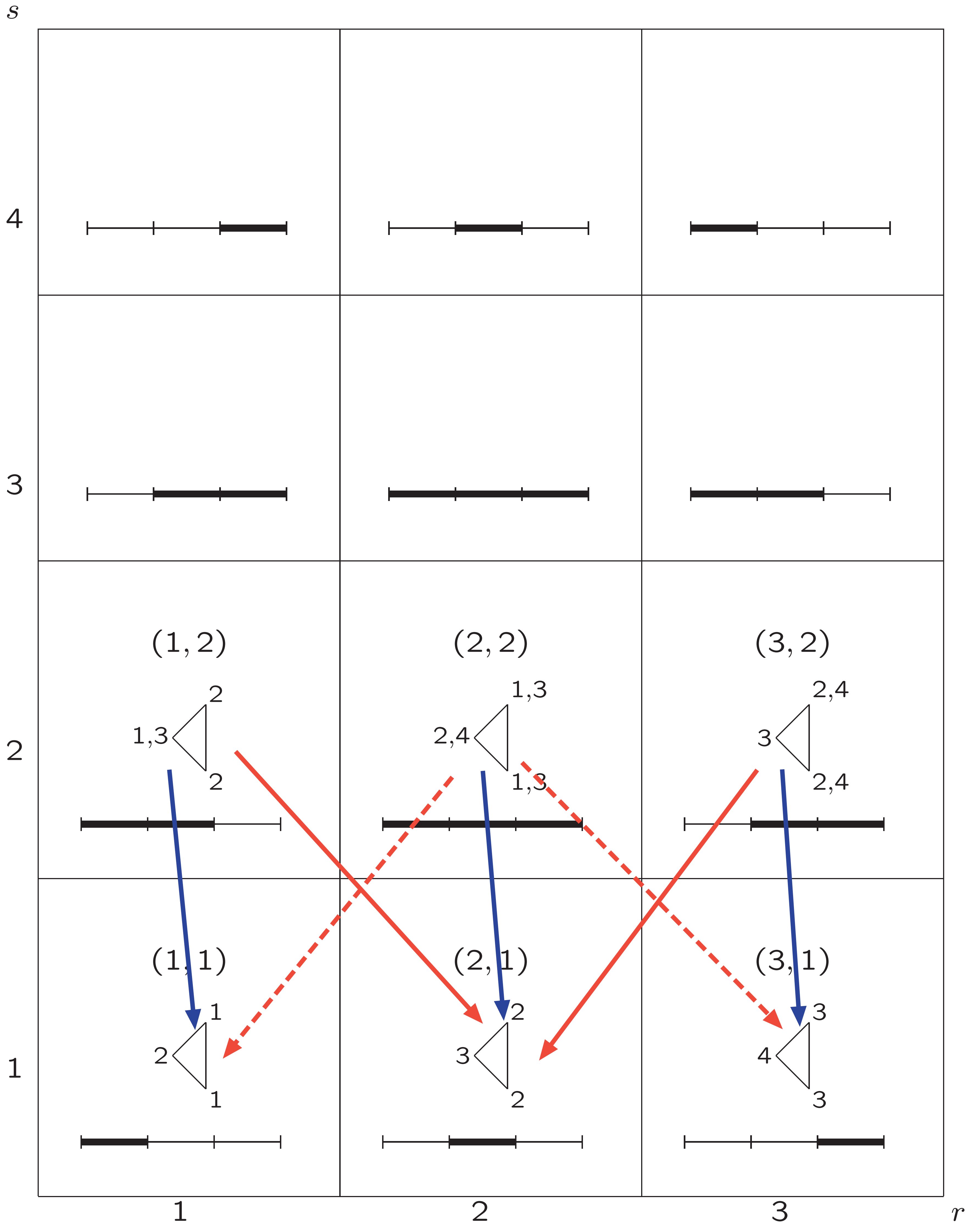
Plus $+$ $(\xi\to 3\pi/2+i\infty)$:

$$B^{(1,1)}\,2\,{}^{1}_{1}(\theta,\xi,1)\to i\tanh\left(\frac{\pi i}{4}-\frac{\theta}{2}\right)=\mathcal{R}^{(1,1)}\,2\,{}^{1}_{1}(\theta)$$

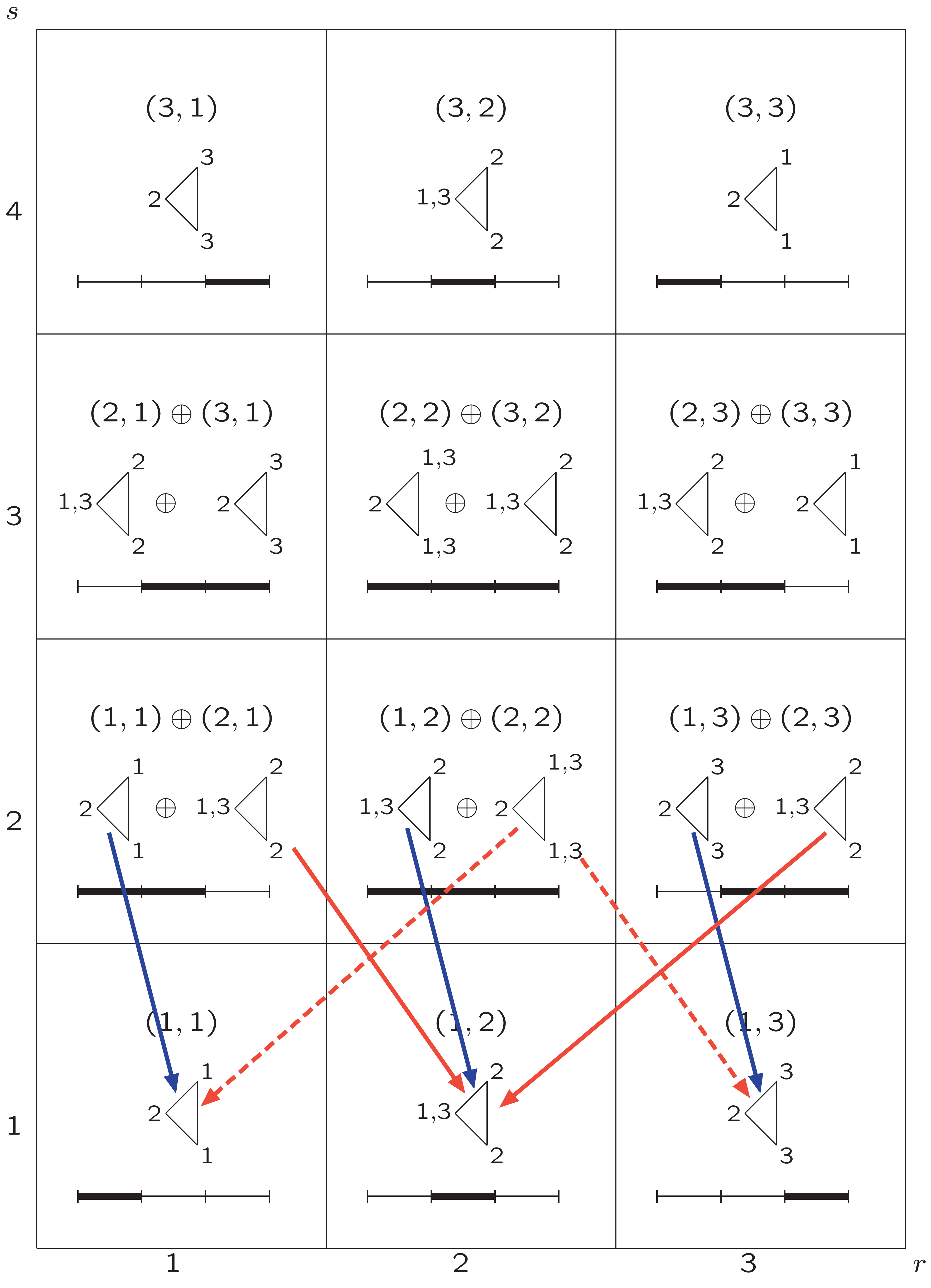
Minus $-$ $(\pi-\xi\to-\pi/2-i\infty)$:

$$B^{(1,2)}\,1\,{}^{2}_{2}(\theta,\pi-\xi,1)\to i\tanh\left(\frac{\pi i}{4}-\frac{\theta}{2}\right)=\mathcal{R}^{(2,1)}\,1\,{}^{2}_{2}(\theta)$$

TIM Conformal Boundary Flows



TIM Massive Boundary Flows



TIM Two-Parameter Flows

- The Tricritical Ising Model admits six two-parameter flows:

$+$ $-$

A_4 A_3 A_4 A_3

Conformal Massive Conformal Massive

$$\begin{tikzcd}
(1,2) \arrow[r, "MR"] \arrow[d, "h"'] & (2,1) \arrow[d, "\xi"] & (1,2) \arrow[r, "MR"] \arrow[d, "h"'] & (1,1) \arrow[d, "\xi"] \\
(2,1) \arrow[r, "MR"] & (1,2) & (1,1) \arrow[r, "MR"] & (1,1)
\end{tikzcd}$$

$$\begin{tikzcd}
(2,2) \arrow[r, "MR"] \arrow[d, "h"'] & (2,2) \arrow[d, "\xi"] & (2,2) \arrow[r, "MR"] \arrow[d, "h"'] & (1,2) \arrow[d, "\xi"] \\
(1,1) \oplus (3,1) \arrow[r, "MR"] & (1,1) \oplus (1,3) & (2,1) \arrow[r, "MR"] & (1,2)
\end{tikzcd}$$

$$\begin{tikzcd}
(3,2) \arrow[r, "MR"] \arrow[d, "h"'] & (2,3) \arrow[d, "\xi"] & (3,2) \arrow[r, "MR"] \arrow[d, "h"'] & (1,3) \arrow[d, "\xi"] \\
(2,1) \arrow[r, "MR"] & (1,2) & (3,1) \arrow[r, "MR"] & (1,3)
\end{tikzcd}$$

Summary

- Explicit expressions have been obtained for the (r, s) boundary S matrices of the massive $\phi_{1,3}$ perturbation of the A_m models.

- The solutions are obtained in the form of direct sums of pairs of "critical" A_{m-1} Behrend-Pearce solutions:

$$\boxed{(r, s) = (s - 1, r) \oplus (s, r) = (s - 1, r) \oplus (m - s, m - r)}$$

- The paired solutions satisfy the boundary bootstrap equation and exhibit the expected $\mathbb{Z}_2$ and Kac table symmetries.

- The vacua that live on the boundary are identified with the allowed edges of the (r, s) conformal boundary conditions as given by the Behrend-Pearce boundary subsets.

- The patterns of massive flows are identical to the patterns of conformal flows given by the Fredenhagen-Schomerus formula.