Excited States of the sine-Gordon model with two boundaries

Changrim Ahn
(Ewha)

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CONTENT

1. XXZ open spin=1/2 chain
2. SG model with two boundaries
3. IR limit: scattering matrices
4. UV limit: c=1 CFT on a strip
5. Future directions

XXZ OPEN CHAIN

$$\mathcal{H} = \frac{1}{2}\Bigg\{ \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh\eta\, \sigma_n^z \sigma_{n+1}^z)$$
$$+ \sinh\eta \big[\coth\alpha_- \tanh\beta_- \sigma_1^z + \operatorname{csch}\alpha_- \operatorname{sech}\beta_- (\cosh\theta_- \sigma_1^x + i \sinh\theta_- \sigma_1^y)$$
$$- \coth\alpha_+ \tanh\beta_+ \sigma_N^z + \operatorname{csch}\alpha_+ \operatorname{sech}\beta_+ (\cosh\theta_+ \sigma_N^x + i \sinh\theta_+ \sigma_N^y)\big]\Bigg\},$$

- **Bethe ansatz solvable if certain conditions are met**
  Nepomechie(2003)

$$\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k$$

k=odd (even) if N= even (odd)

- **Dirichlet limit**

$$\beta_\pm \to \pm\infty$$

- **Energy**

$$E = \sinh^2\eta \sum_{j=1}^{M} \frac{1}{\sinh(\tilde{u}_j - \frac{\eta}{2})\sinh(\tilde{u}_j + \frac{\eta}{2})}$$

$$+ \frac{1}{2}\sinh\eta(\coth\alpha_- + \tanh\beta_- + \coth\alpha_+ + \tanh\beta_+) + \frac{1}{2}(N-1)\cosh\eta.$$

•**Bethe ansatz Eq.**

$$\left(\frac{\sinh(\tilde{u}_j + \frac{\eta}{2})}{\sinh(\tilde{u}_j - \frac{\eta}{2})}\right)^{2N} \frac{\sinh(\tilde{2}u_j + \eta)}{\sinh(\tilde{2}u_j - \eta)} \frac{\sinh(\tilde{u}_j - \frac{\eta}{2} + \alpha_-)}{\sinh(\tilde{u}_j + \frac{\eta}{2} - \alpha_-)} \frac{\cosh(\tilde{u}_j - \frac{\eta}{2} + \beta_-)}{\cosh(\tilde{u}_j + \frac{\eta}{2} - \beta_-)}$$

$$\times \frac{\sinh(\tilde{u}_j - \frac{\eta}{2} + \alpha_+)}{\sinh(\tilde{u}_j + \frac{\eta}{2} - \alpha_+)} \frac{\cosh(\tilde{u}_j - \frac{\eta}{2} + \beta_+)}{\cosh(\tilde{u}_j + \frac{\eta}{2} - \beta_+)}$$

$$= -\prod_{k=1}^{M} \frac{\sinh(\tilde{u}_j - \tilde{u}_k + \eta)}{\sinh(\tilde{u}_j - \tilde{u}_k - \eta)} \frac{\sinh(\tilde{u}_j + \tilde{u}_k + \eta)}{\sinh(\tilde{u}_j + \tilde{u}_k - \eta)}, \qquad M = \frac{1}{2}(N-1+k).$$

- **Constraint Eq.**

$$\eta = i\mu, \qquad \alpha_\pm = i\mu a_\pm, \qquad \beta_\pm = \mu b_\pm, \qquad \theta_\pm = i\mu c_\pm.$$

$$a_- + a_+ = \pm|c_- - c_+| + k$$
$$b_- + b_+ = 0.$$

$$\frac{1}{2} - \nu < a_\pm < \frac{1}{2} + \nu$$

$$e_1(\lambda_j)^{2N+1} g_1(\lambda_j) \frac{e_{2a_- -1}(\lambda_j) e_{2a_+ -1}(\lambda_j)}{g_{1+2ib_-}(\lambda_j) g_{1+2ib_+}(\lambda_j)} = -\prod_{k=1}^{M} e_2(\lambda_j - \lambda_k) e_2(\lambda_j + \lambda_k)$$

$$e_n(\lambda) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})} \qquad g_n(\lambda) = e_n\left(\lambda \pm \frac{i\pi}{2\mu}\right) = \frac{\cosh \mu(\lambda + \frac{in}{2})}{\cosh \mu(\lambda - \frac{in}{2})}$$

$$h(\lambda) = \frac{1}{2\pi}\Big\{(2N+1) q_1(\lambda) + r_1(\lambda) + q_{2a_- -1}(\lambda) - r_{1+2ib_-}(\lambda) + q_{2a_+ -1}(\lambda) - r_{1+2ib_+}(\lambda) - \sum_{k=1}^{N/2} [q_2(\lambda - \lambda_k) + q_2(\lambda + \lambda_k)]\Big\}$$

$$q_n(\lambda) = \pi + i \ln e_n(\lambda) = 2 \tan^{-1}\big(\cot(n\mu/2) \tanh(\mu\lambda)\big)$$

$$r_n(\lambda) = i \ln g_n(\lambda).$$

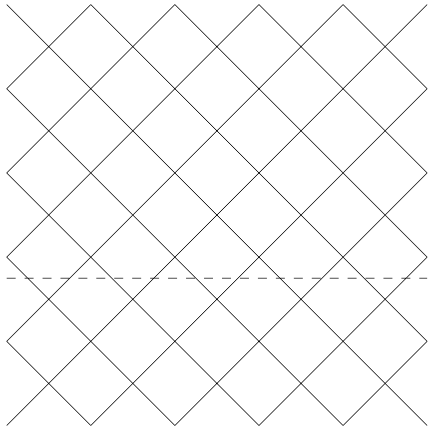
$$h(\lambda_j) = j, \quad j = 1, \ldots, \frac{N}{2}$$

$$a_n(\lambda) = \frac{1}{2\pi}\frac{d}{d\lambda} q_n(\lambda) = \frac{\mu}{\pi} \frac{\sin(n\mu)}{\cosh(2\mu\lambda) - \cos(n\mu)},$$

$$b_n(\lambda) = \frac{1}{2\pi}\frac{d}{d\lambda} r_n(\lambda) = -\frac{\mu}{\pi} \frac{\sin(n\mu)}{\cosh(2\mu\lambda) + \cos(n\mu)}$$

INHOMOGENEITY

- XXZ SPIN CHAIN WITH ALTERNATING SPECTRAL PARAMETERS



- Bethe ansatz can be written as

$$h(\lambda) = \frac{1}{2\pi}\Bigg\{N\big[q_1(\lambda+\Lambda)+q_1(\lambda-\Lambda)\big]+q_1(\lambda)+r_1(\lambda)+q_{2a_--1}(\lambda)-r_{1+2ib_-}(\lambda)+ q_{2a_+-1}(\lambda)-r_{1+2ib_+}(\lambda)-\sum_{k=1}^{N/2}\big[q_2(\lambda-\lambda_k)+q_2(\lambda+\lambda_k)\big]\Bigg\}$$

$$E=-\frac{1}{\Delta}\sum_{j=1}^{N/2}\big[a_1(\lambda_j+\Lambda)+a_1(\lambda_j-\Lambda)\big]=-\frac{1}{\Delta}\Bigg\{N\int_{-\infty}^{\infty}d\lambda\, a_1(\Lambda-\lambda)\rho(\lambda)-a_1(\Lambda)\Bigg\}$$

Bulk and Boundary Energies

$$E_{\text{bulk}} = \frac{1}{4} m^2 R \cot(\nu\pi/2)$$

$$E^{\pm}_{\text{boundary}} = -\frac{m}{2}\left[-\frac{1}{2}\cot(\nu\pi/4) - \frac{1}{2} + \frac{\cos((\nu - 2s_{\pm}a_{\pm})\pi/2)}{\sin(\nu\pi/2)} + \frac{\cosh(\pi b_{\pm})}{\sin(\nu\pi/2)} \right]$$

Continuum Limit

$$R = N\Delta, \qquad m = \frac{2}{\Delta} e^{-\pi\Lambda}$$

SG model with two boundaries

$$\mathcal{A} = \int_{-\infty}^{\infty} dy \int_{x_-}^{x_+} dx\, A(\varphi, \partial_\mu \varphi) + \int_{-\infty}^{\infty} dy \left[B_-\left(\varphi, \frac{d\varphi}{dy}\right)\Big|_{x=x_-} + B_+\left(\varphi, \frac{d\varphi}{dy}\right)\Big|_{x=x_+} \right]$$

$$B_\pm(\varphi, \partial_y \varphi) = -\mu_\pm \cos\left(\frac{\beta}{2}(\varphi - \varphi_0^\pm)\right) \pm i\kappa_\pm \partial_y \varphi$$

- **Boundary S-matrix**

$$R(\theta; \eta, \vartheta, \gamma) = r_0(\theta) r_1(\theta; \eta, \vartheta) M(\theta; \eta, \vartheta, \gamma) \qquad M(\theta; \eta, \vartheta, \gamma) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$m_{11} = \cos\eta \cosh\vartheta \cosh(\lambda\theta) + i \sin\eta \sinh\vartheta \sinh(\lambda\theta),$$

$$m_{22} = \cos\eta \cosh\vartheta \cosh(\lambda\theta) - i \sin\eta \sinh\vartheta \sinh(\lambda\theta),$$

$$m_{12} = i e^{i\gamma} \sinh(\lambda\theta) \cosh(\lambda\theta),$$

$$m_{21} = i e^{-i\gamma} \sinh(\lambda\theta) \cosh(\lambda\theta).$$

Relations of boundary parameters

- **Bulk energies:**

$$\lambda \equiv \frac{8\pi}{\beta^2} - 1 = \frac{1}{\nu - 1}$$

- **Boundary energies:**

$$E(\eta,\vartheta) = -\frac{m}{2\cos(\pi/(2\lambda))}\left[-\frac{1}{2}\cos\big(\pi/(2\lambda)\big) + \frac{1}{2}\sin\big(\pi/(2\lambda)\big) - \frac{1}{2} + \cos(\eta/\lambda) + \cosh(\vartheta/\lambda)\right]$$

$$\eta_\pm = (1 + \lambda - 2\lambda a_\pm)\frac{\pi}{2}$$

$$\vartheta_\pm = \lambda\pi b_\pm$$

$$\cos\left(\frac{\beta^2}{8\pi}(\eta_\pm + i\vartheta_\pm)\right) = \frac{\mu_\pm}{\mu_c} e^{\mp\frac{i}{2}\beta\varphi_0^\pm}$$

- **Periodicity argument (next):**

$$\gamma_\pm = \frac{4\pi}{\beta}\kappa_\pm$$

- **Path integral of boundary SG action**

$$\int \mathcal{D}\varphi e^{-\mathcal{A}_E} \longrightarrow e^{-i(\kappa_+ \Delta\varphi(x_+) - \kappa_- \Delta\varphi(x_-))} \int \mathcal{D}\varphi e^{-\mathcal{A}_E}$$

$$\Delta\varphi(x) \equiv \int_{-\infty}^{\infty} dy\, \partial_y \varphi(x,y) = \varphi(x, y=\infty) - \varphi(x, y=-\infty)$$

- **Periodicity**

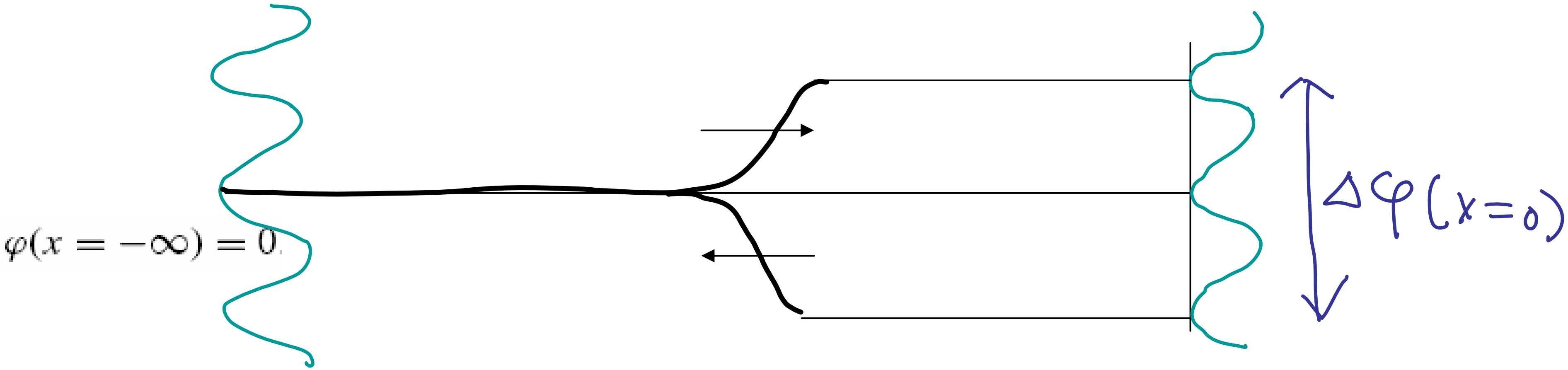
$$\varphi(x,y) \mapsto \varphi(x,y) + \frac{4\pi}{\beta} \longrightarrow \Delta\varphi(x_\pm) = \frac{4\pi}{\beta} n_\pm$$

$$e^{-\frac{4\pi i}{\beta}(\kappa_+ n_+ - \kappa_- n_-)}$$

- **Periodicity of kappa**

$$\kappa_\pm \mapsto \kappa_\pm + \frac{\beta}{2}$$

- **SCATTERING ON ONE BOUNDARY:**



$$\int \mathcal{D}\varphi \exp\left(i \int_{-\infty}^{\infty} dt\, L_M \right) = \boxed{e^{-i\kappa_+ \Delta\varphi}} \int \mathcal{D}\varphi \exp\left(i \int_{-\infty}^{\infty} dt\, L_M(\kappa_+ = 0) \right)$$

$$\Delta\varphi = \varphi(x=0, t=\infty) - \varphi(x=0, t=-\infty).$$

- **SOLITON TO SOLITON:** $\varphi(x=0, t=\mp\infty) = \frac{2\pi}{\beta} \quad \Rightarrow \quad \Delta\varphi = 0$

- **SOLITON TO ANTI-SOLITON:** $\varphi(x=0, t=\mp\infty) = \pm\frac{2\pi}{\beta} \quad \Rightarrow \quad \Delta\varphi = -\frac{4\pi}{\beta}$

Excited states

- **Counting function:**

$$h^{(+)}(\lambda) = \frac{1}{2\pi}\left\{ N\left[q_1(\lambda+\Lambda)+q_1(\lambda-\Lambda)\right] + q_1(\lambda) + r_1(\lambda) + q_{2a_- -1}(\lambda) - r_{1+2ib_-}(\lambda) + q_{2a_+ -1}(\lambda) - r_{1+2ib_+}(\lambda) - \sum_{k=1}^{M^{(+)}}\left[q_2(\lambda-\lambda_k)+q_2(\lambda+\lambda_k)\right]\right\}$$

$$M^{(+)} = \frac{1}{2}(N-1+k)$$

- **BAE:** $h^{(+)}(\lambda_j) = I_j^{(+)}, \quad j = 1, \ldots, M^{(+)}$

- **Other states are from**

$$h^{(+)}(\lambda) \longrightarrow h^{(-)}(\lambda)$$

$$(a_\pm, b_\pm) \mapsto (-a_\pm, -b_\pm)$$

Excited states

- Lattice counting equation:

$$N_H - 2N_S = M_C + 2M_W\,\mathrm{step}(\nu - 2) + 1 + \frac{1}{2}(s_+ + s_-) - k + \left\lfloor \frac{1}{2} - \frac{1}{\nu}(a_+ + a_- - k) \right\rfloor$$

- Ex: no holes, complex roots, … for k=1

$$0 = \frac{1}{2}(s_+ + s_-) + \left\lfloor \frac{1}{2} - \frac{1}{\nu}(a_+ + a_- - 1) \right\rfloor$$

- Continuum counting equation:

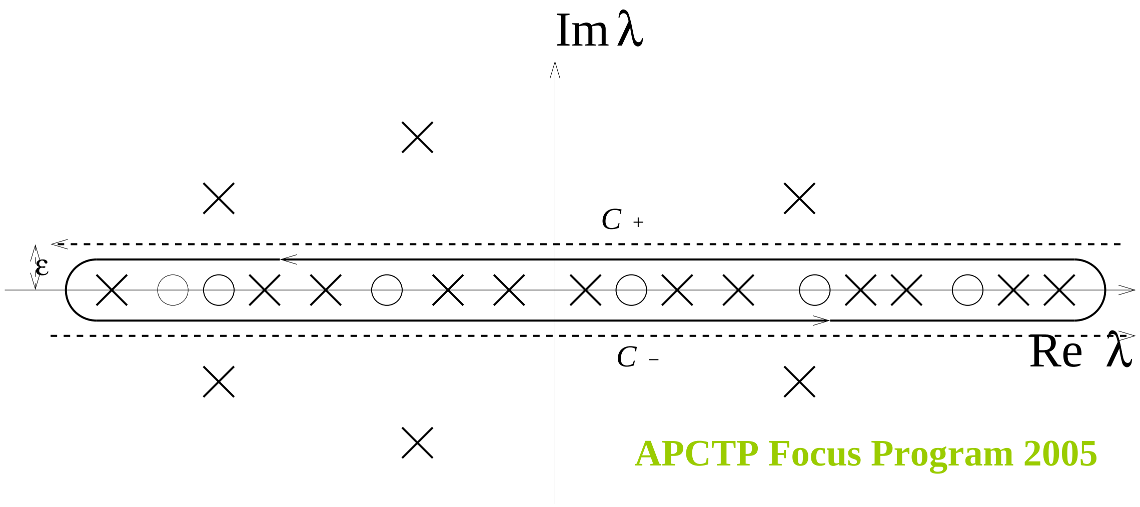
$$N_H - 2N_S = M_C + 2M_W\,\mathrm{step}(\nu - 2) + 1 + \frac{1}{2}(s_+ + s_-) - k$$

Nonlinear Integral Equation

$$f^{(+)}(\theta) = 2imL\sinh\theta + iP^{(+)}_{\text{bdry}}(\theta) + ig(\theta) \;\; + \frac{2i}{\pi}\int_{-\infty}^{\infty} d\theta' \,\Im\mathrm{m}\, G(\theta-\theta'-i\epsilon)\ln\left(1-e^{f^{(+)}(\theta'+i\epsilon)}\right)$$

$$g(\theta) = \sum_{j=1}^{N_H}\left[\chi(\theta-\theta_j^H)+\chi(\theta+\theta_j^H)\right] - \sum_{j=1}^{N_S}\left[\chi(\theta-\theta_j^S+i\epsilon)+\chi(\theta-\theta_j^S-i\epsilon)+\chi(\theta+\theta_j^S+i\epsilon)+\chi(\theta+\theta_j^S-i\epsilon)\right]$$

$$E_{\text{Casimir}} = m\sum_{j=1}^{N_H}\cosh\theta_j^H - m\sum_{j=1}^{N_S}\left[\cosh(\theta_j^S+i\epsilon)+\cosh(\theta_j^S-i\epsilon)\right] - \frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta \,\Im\mathrm{m}\sinh(\theta+i\epsilon)\ln\left(1-e^{f^{(+)}(\theta+i\epsilon)}\right)$$



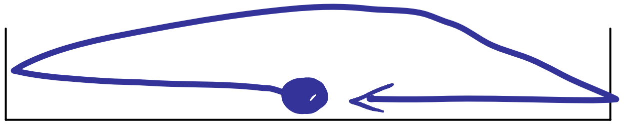
IR limit

- **Large L with 1 hole:** k=1

$$f^{(+)}(\theta) = 2imL\sinh\theta + i\,P^{(+)}_{\rm bdry}(\theta) + i\,\chi(\theta-\theta_H) + i\,\chi(\theta+\theta_H)$$

$$e^{2imL\sinh\theta_H}\, e^{i(P^{(+)}_{\rm bdry}(\theta_H)+\chi(2\theta_H))} = 1$$

- **Yang equation**



$$e^{2imL\sinh\theta_H}\, R(\theta_H;\eta_+,\vartheta_+,\gamma_+)\, R(\theta_H;\eta_-,\vartheta_-,\gamma_-)\big|\theta_H,(\pm)\big\rangle = \big|\theta_H,(\pm)\big\rangle$$

$$e^{i(P^{(\pm)}_{\rm bdry}(\theta)+\chi(2\theta))} = r_0(\theta)^2 r_1(\theta;\eta_+,\vartheta_+) r_1(\theta;\eta_-,\vartheta_-)\Lambda^{(\pm)}$$

if constraint eqs. Are satisfied

UV limit $mL \to 0$

- consider only holes & specials and rescale

$$N_H = N_H^0 + N_H^\infty, \qquad N_S = N_S^0 + N_S^\infty$$

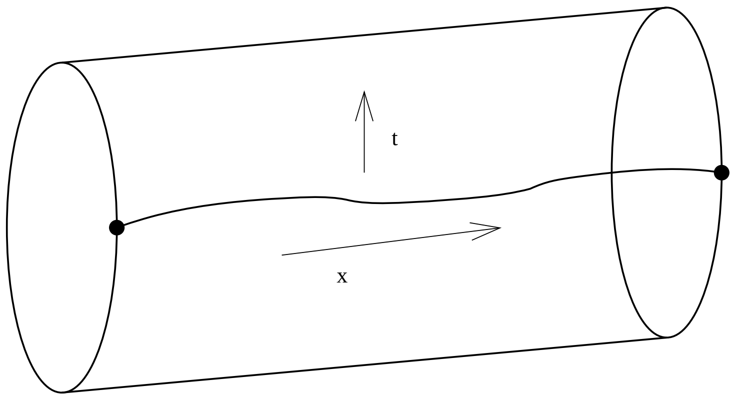
$$\theta = \ln\frac{1}{mL} + \tilde{\theta} \qquad \theta_j^{H\infty} = \ln\frac{1}{mL} + \tilde{\theta}_j^{H\infty} \qquad \theta_j^{S\infty} = \ln\frac{1}{mL} + \tilde{\theta}_j^{S\infty}$$

- Standard UV calculations lead to

$$\Delta_n = \frac{1}{4\nu(\nu-1)}\left\{\nu\left[\frac{1}{2}(s_+ + s_-) + N_H^0 - N_H^\infty - 2N_S^0 + 2N_S^\infty\right] - \left(k - 1 + 2N_H^0 - 4N_S^0\right) \mp |c_- - c_+|\right\}^2 + \sum_{j=1}^{N_H^\infty} I_j^{H\infty} - 2\sum_{j=1}^{N_S^\infty} I_j^{S\infty} - \frac{1}{2}\left(N_H^\infty - 2N_S^\infty\right)\left(N_H^\infty - 2N_S^\infty + 1\right)$$

c=1 free boson on a strip

- **Conformal BC**



- **Bulk Dimension**

$$\Delta = \frac{1}{2\pi}\left(\frac{k}{2r} + \pi\omega r\right)^2$$

- **Compactified boson**

$$\phi \to \phi + 2\pi r$$

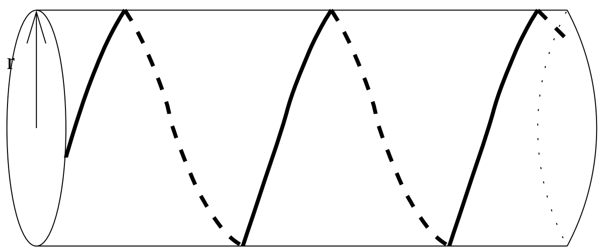
Conformal Boundary Conditions

1. Dirichlet BC

$$\phi(0) = \phi_0, \quad \phi(L) = \phi_0'$$

– Boundary state $$|B_D(\phi_0)\rangle \propto \sum_{-\infty}^{\infty} e^{-ik\phi_0/r} e^{-\sum \frac{1}{n} a_{-n}\bar{a}_{-n}} |0, k\rangle$$

– Partition function



$$Z = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} q^{\frac{1}{2\pi}(2\pi r n + \phi_0 - \phi_0')^2}$$

2. Neumann BC: Dirichlet of the dual boson

A New BC

- **Boundary action**

$$S = \int d^2x \frac{1}{2}(\partial_\mu \phi)^2 + \int dt \left[-\kappa_+ \dot{\phi}(0) + \kappa_- \dot{\phi}(L) \right]$$

- **Partition function**

$$Z = \int \mathcal{D}\phi e^{iS} = e^{-i\kappa_+ \Delta\phi(0) + i\kappa_- \Delta\phi(L)} Z_0$$

- **Periodic condition in time direction**

$$\Delta\phi(x_\pm) = \phi(x_\pm, T) - \phi(x_\pm, 0) = 2\pi r n_\pm$$

- **Periodicity:** $\kappa_\pm \to \kappa_\pm + \frac{1}{r}$

Canonical quantization

- **Mode expansion**

$$\phi(x,t) = \phi_0 + p_0 \frac{t}{L} + \frac{i}{\sqrt{\pi}} \sum_{n \neq 0} a_n \cos \frac{\pi n (x-L)}{L} e^{-in\pi t/L}$$

- **Conjugate momentum**

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial_t \phi - \kappa_+ \delta(x) + \kappa_- \delta(x-L)$$

- **Zero momentum** $\Pi_0 = p_0 - \kappa_+ + \kappa_-$

- **Commutation relation** $[\Pi_0, \phi_0] = -i$

- **Hamiltonian**

$$H = \frac{1}{2L} p_0^2 + \text{modes}$$

- **Wave function and momentum quantization**

$$\Psi(\phi_0) = e^{i\Pi_0\phi_0} \qquad \Pi_0 = \frac{n}{r}$$

- **Energy and conformal dimension**

$$E = \frac{1}{2L}\left(\frac{n}{r} + \kappa_+ - \kappa_-\right)^2 \quad \rightarrow \quad \boxed{\Delta = \frac{1}{2\pi}\left(\frac{n}{r} + \kappa_+ - \kappa_-\right)^2}$$

- **Dual to Dirichlet BC:** winding to (shifted) momentum
- **Consistent with periodicity of kappa**
- **Same as UV limit if** $k - 1 + 2N_H^0 - 4N_S^0 = 2n$

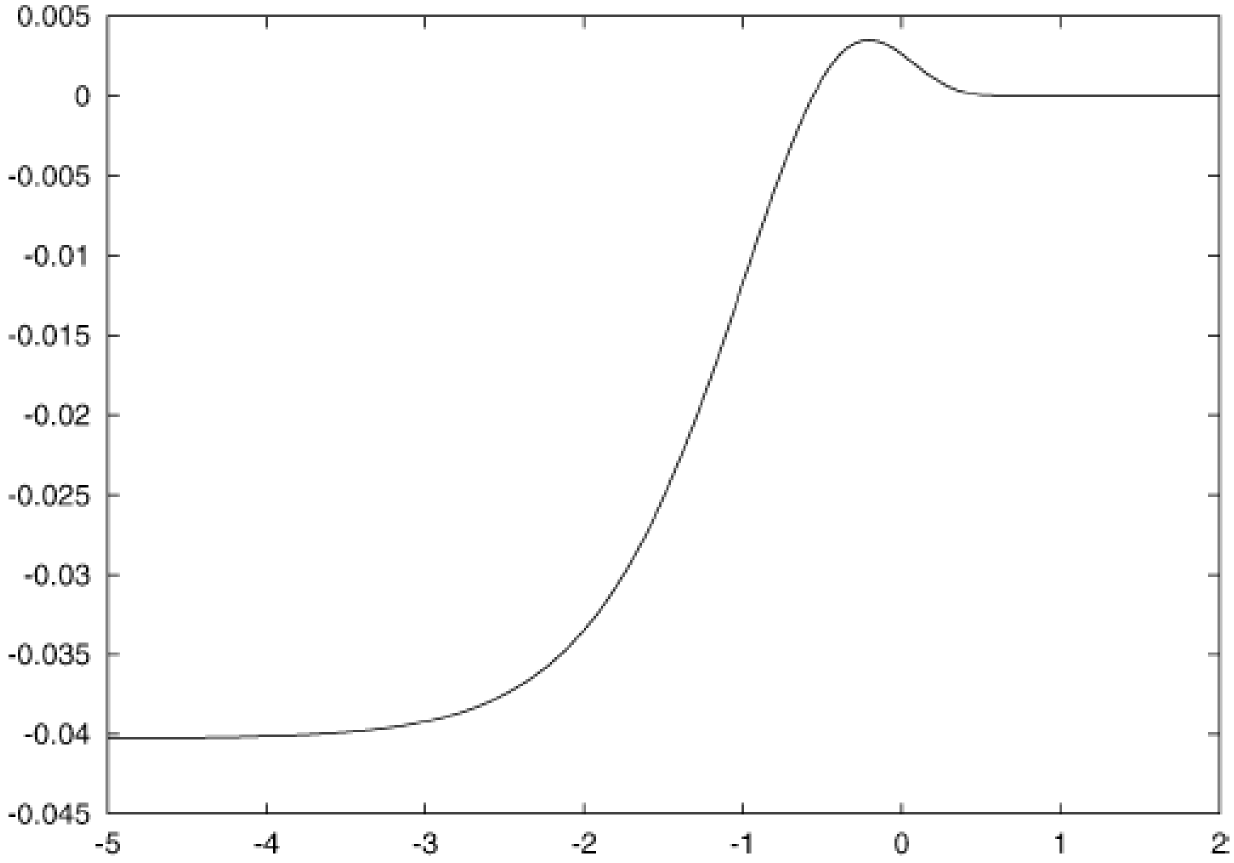
Numerical analysis: ground state

- **Normalized Casimir energy**

$$\mathcal{E} = \frac{L E_{\text{Casimir}}}{\pi + mL} = -\frac{\pi c_{\text{eff}}}{24(\pi + mL)}$$

$$\mathcal{E} \to \Delta_n - \frac{1}{24} \quad \text{for } L \to 0$$

$$\mathcal{E} \to N_H \quad \text{for } L \to \infty.$$



- **Near UV region**

$$E_{\text{NLIE}}(L) = -\epsilon_{\text{bulk}} L - \epsilon_{\text{boundary}} + \frac{\pi}{L}\left(E_{|0\rangle} - \frac{1}{24} + c_2^0\left(\frac{\pi}{L}\right)^{2(\Delta-1)} + c_4^0\left(\frac{\pi}{L}\right)^{4(\Delta-1)} + c_6^0\left(\frac{\pi}{L}\right)^{6(\Delta-1)} + \cdots\right)$$

- **Boundary CFT vs. TCSA**

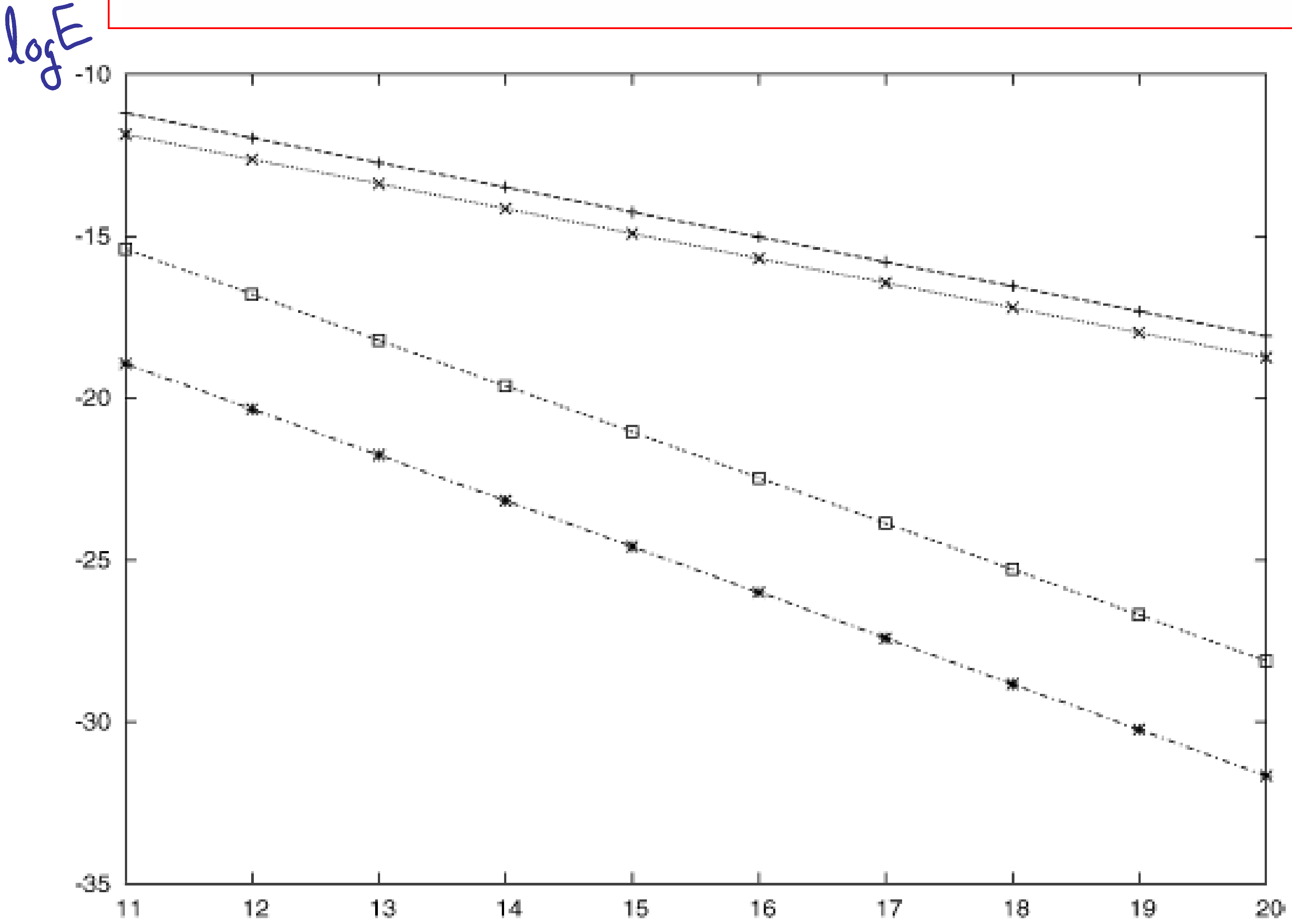
Table 1
Comparison of NLIE and BCPT results for $c_2^0 m^{2(\Delta-1)}$, for various values of bulk coupling constant ν and for boundary parameter values $a_+ = 1.8$, $a_- = -0.9$, and $b_+ = -b_- = 0.41444$

ν	NLIE $c_2^0 m^{2(\Delta-1)}$	BCPT $c_2^0 m^{2(\Delta-1)}$
1.70	−5.3215286975	−5.3215288274
1.80	−7.4632436186	−7.4632435914
1.93	−19.5148929102	−19.5148929079
2.20	5.6819407377	5.6819407318
2.40	2.4879276564	2.4879276494
2.60	1.4601870563	1.4601870411

(1.70–1.93: attractive; 2.20–2.60: repulsive)

- **Near IR region: Luscher correction due to a fundamental breather**

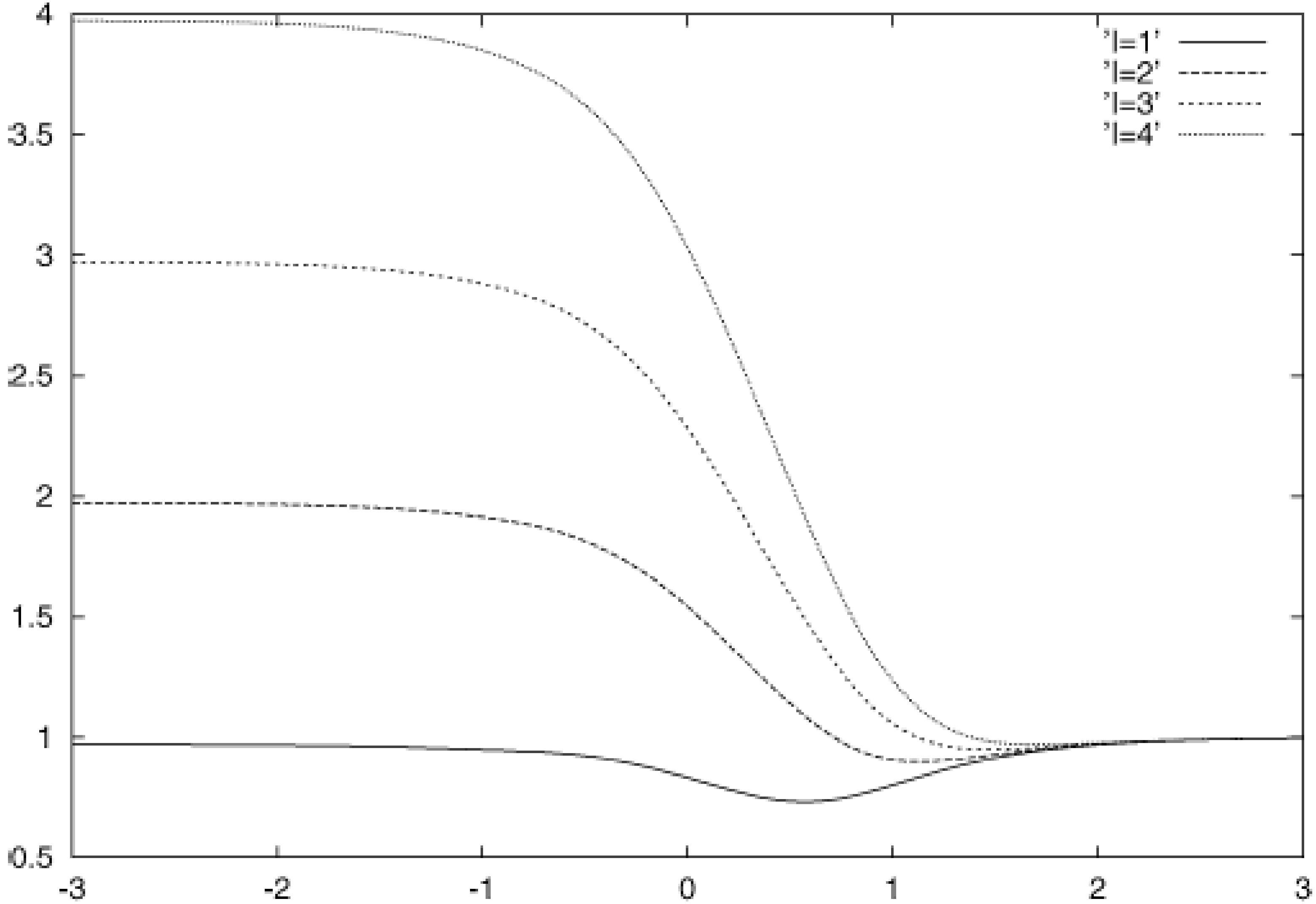
$$E_{\rm NLIE}(L) = m_1 \frac{1+\cos\frac{\pi}{2\lambda} - \sin\frac{\pi}{2\lambda}}{1-\cos\frac{\pi}{2\lambda}+\sin\frac{\pi}{2\lambda}} \tan\frac{\eta_+}{2\lambda}\tanh\frac{\vartheta_+}{2\lambda}\tan\frac{\eta_-}{2\lambda}\tanh\frac{\vartheta_-}{2\lambda} e^{-m_1 L} + \cdots$$



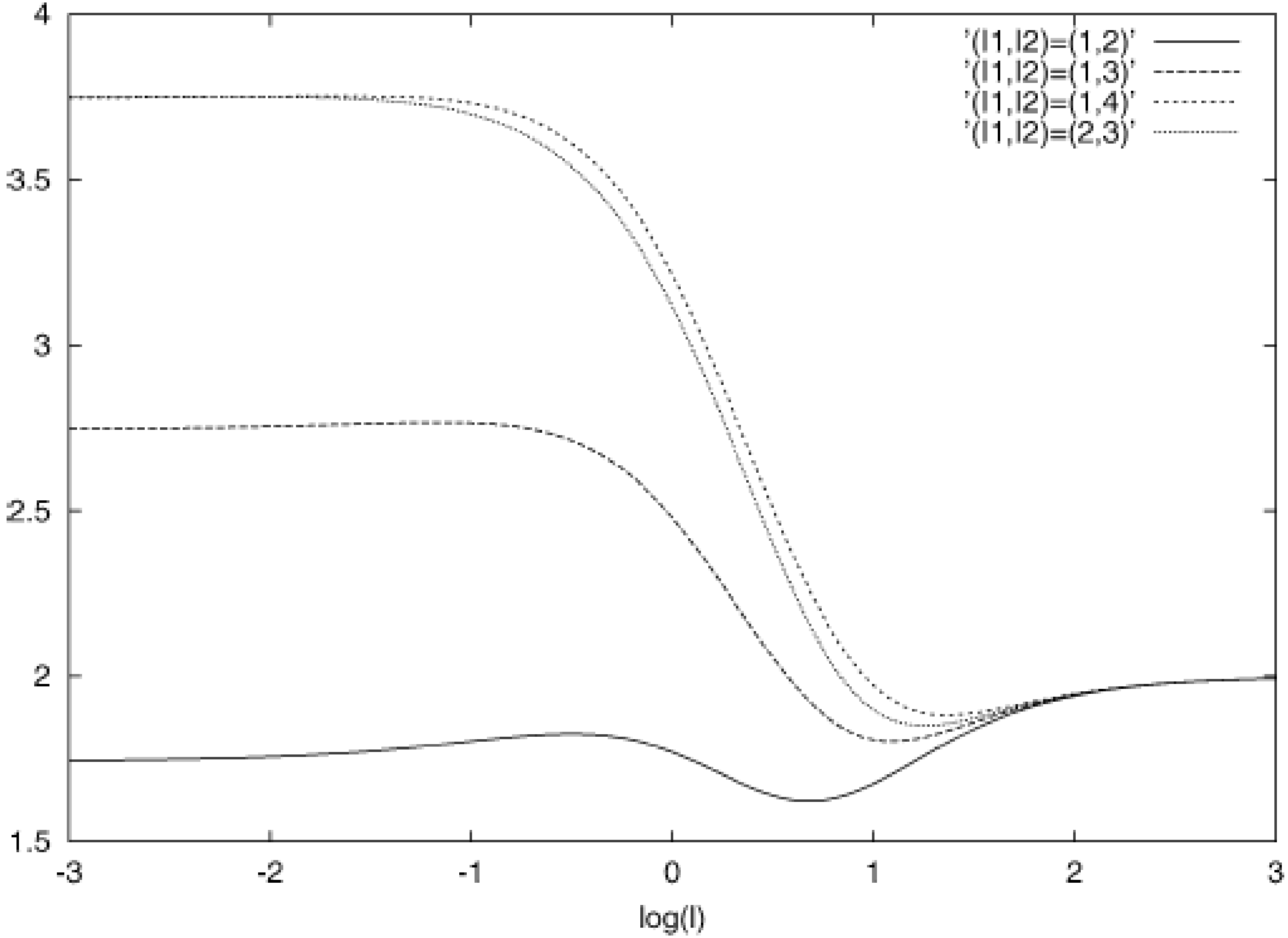
$$m_1 = 2m \sin(\frac{\pi}{2\lambda})$$

Numerical analysis: excited states

- Normalized Casimir energy of 1 hole state



- **Normalized Casimir energy of 2 hole state**



Future directions

- Other bulk excited states
- Boundary excited states
- Removing constraints (rafael's talk)
- Extension to super sine-gordon models
- applications